

VOEVODSKY'S MOTIVES AND WEIL RECIPROCITY

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ABSTRACT. We describe Somekawa's K -group associated to a finite collection of semi-abelian varieties (or more general sheaves) in terms of the tensor product in Voevodsky's category of motives. While Somekawa's definition is based on Weil reciprocity, Voevodsky's category is based on homotopy invariance. We apply this to explicit descriptions of certain algebraic cycles.

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1. INTRODUCTION

1.1. In this article, we construct an isomorphism

$$(1.1) \quad K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{DM}_{-}^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]).$$

Here k is a perfect field, and $\mathcal{F}_1, \dots, \mathcal{F}_n$ are homotopy invariant Nisnevich sheaves with transfers in the sense of [32]. On the right hand side, the tensor product $\mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]$ is computed in Voevodsky's triangulated category $\mathbf{DM}_{-}^{\mathrm{eff}}$ of effective motivic complexes. The group $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ will be defined in Definition 5.1 by an explicit set of generators and relations: it is a generalization of the group which was defined by K. Kato and studied by M. Somekawa in [24] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semi-abelian varieties.

1.2. In the introduction of [24], Somekawa wrote that he expected an isomorphism of the form

$$K(k; G_1, \dots, G_n) \simeq \mathrm{Ext}_{MM}^n(\mathbf{Z}, G_1[-1] \otimes \cdots \otimes G_n[-1])$$

where MM is a conjectural abelian category of mixed motives over k , G_1, \dots, G_n are semi-abelian varieties over k , and $G_1[-1], \dots, G_n[-1]$ are the corresponding 1-motives. Since we do not have such a category MM at hand, (1.1) provides the closest approximation to Somekawa's expectation.

1.3. The most basic case of (1.1) is $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbf{G}_m$. By [24, Theorem 1.4], the left hand side is isomorphic to the usual Milnor K -group $K_n^M(k)$. The right hand side is almost by definition the motivic cohomology group $H^n(k, \mathbf{Z}(n))$. Thus, when k is perfect, we get a new and less combinatorial proof of the Suslin-Voevodsky isomorphism [28, Thm. 3.4], [16, Thm. 5.1]

$$(1.2) \quad K_n^M(k) \simeq H^n(k, \mathbf{Z}(n)).$$

1.4. The isomorphism (1.1) also has the following application to algebraic cycles. Let X be a k -scheme of finite type. Write $\underline{CH}_0(X)$ for the homotopy invariant Nisnevich sheaf with transfers

$$U \mapsto CH_0(X \times_k k(U)) \quad (U \text{ smooth connected})$$

see [7, Th. 2.2]. Let $i, j \in \mathbf{Z}$. We write $CH_i(X, j)$ for Bloch's homological higher Chow group [14, 1.1]: if X is equidimensional of dimension d , it agrees with the group $CH^{d-i}(X, j)$ of [4].

1.5. Theorem. *Suppose $\mathrm{char} k = 0$. Let X_1, \dots, X_n be quasi-projective k -schemes. Put $X = X_1 \times \cdots \times X_n$. For any $r \geq 0$, we have an*

isomorphism

$$(1.3) \quad K(k; \underline{CH}_0(X_1), \dots, \underline{CH}_0(X_n), \mathbf{G}_m, \dots, \mathbf{G}_m) \xrightarrow{\sim} CH_{-r}(X, r),$$

where we put r copies of \mathbf{G}_m on the left hand side.¹

1.6. When X_1, \dots, X_n are smooth projective,² special cases of (1.3) were previously known. The case $r = 0$ was proved by Raskind and Spieß [20, Corollary 4.2.6], and the case $n = 1$ was proved by Akhtar [1, Theorem 6.1] (without assuming k to be perfect). The extension to non smooth projective varieties is new and nontrivial.

1.7. Theorem 1.5 is proven using the Borel-Moore motivic homology introduced in [6, §9]. We also have a variant which involves motivic homology, see Theorem 12.3. Here is an application. Let C_1, C_2 be two smooth connected curves over our perfect field k , and put $S = C_1 \times C_2$. Assume that C_1 and C_2 both have a 0-cycle of degree 1. Then the special case $n = 2, r = 0$ of Theorem 12.3 gives an isomorphism

$$\mathbf{Z} \oplus \text{Alb}_S(k) \oplus K(k; \text{Alb}_{C_1}, \text{Alb}_{C_2}) \xrightarrow{\sim} H_0(S, \mathbf{Z}).$$

Here, for a smooth variety X , we denote by Alb_X the Albanese variety of X in the sense of Serre [22]; it is a semi-abelian variety universal for morphisms from X to semi-abelian varieties (compare [32, Th. 3.4.2]). The right hand side in this case is Suslin homology [29], see §12.2.

Since Somekawa's groups are defined in an explicit manner, one can sometimes determine the structure of $K(k; \text{Alb}_{C_1}, \text{Alb}_{C_2})$ completely. For instance, when k is finite, we have $K(k; \text{Alb}_{C_1}, \text{Alb}_{C_2}) = 0$ by [9]. This immediately implies the bijectivity of the generalized Albanese map

$$a_S : H_0(S, \mathbf{Z})^{\text{deg}=0} \rightarrow \text{Alb}_S(k)$$

of Ramachandran and Spieß-Szamuely [25]. Note that a_S is not bijective for a smooth projective surface S in general, see [12, Prop. 9].

1.8. We conclude this introduction by pointing out the main difficulty and main ideas in the proof of (1.1).

The definitions of the two sides of (1.1) are quite different: the left hand side is based on *Weil reciprocity*, while the right hand side is based on *homotopy invariance*. Thus it is not even obvious how to define a map (1.1) to start with. Our solution is to write both sides as quotients of a common larger group, and to prove that one quotient factors

¹Using recent results of Shane Kelly [13], one may remove the characteristic zero hypothesis if we invert the exponential characteristic of k .

²In this case, Theorem 1.5 is valid in any characteristic, see Remark 12.4.

through the other. This provides a map (1.1) which is automatically surjective (Theorem 5.3).

The proof of its injectivity turns out to be much more difficult. We need to find many relations coming from Weil reciprocity. Our main idea, inspired by [24, Theorem 1.4] (recalled in §1.3), is to use the *Steinberg relation* to create Weil reciprocity relations. To show that this provides us with enough such relations, we need to carry out a heavy computation of symbols in §11.

Acknowledgements. Work in this direction had been done previously by Mochizuki [17]. The surjective map (1.1) was announced in [26, Remark 10 (b)]. This research was started by the first author, who wrote the first part of this paper [11]. The collaboration began when the second author visited the Institute of Mathematics of Jussieu in October 2010. Somehow, the research accelerated after the earthquake on March 11, 2011 in Japan. We wish to acknowledge the pleasure of such a fruitful collaboration, along these circumstances. We also thank the referees for a very careful reading of this manuscript, and Frédéric Déglise for requesting an explicit proof of (2.4).

We acknowledge the depth of the ideas of Milnor, Kato, Somekawa, Suslin and Voevodsky. Especially we are impressed by the relevance of the Steinberg relation in this story.

2. MACKEY FUNCTORS AND PRESHEAVES WITH TRANSFERS

2.1. A *Mackey functor* over k is a contravariant additive (i.e., commuting with coproducts) functor A from the category of étale k -schemes to the category of abelian groups, provided with a covariant structure verifying the following exchange condition: if

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ g' \downarrow & & g \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

is a cartesian square of étale k -schemes, then the diagram

$$\begin{array}{ccc} A(Y') & \xrightarrow{f'^*} & A(Y) \\ g'_* \downarrow & & g_* \downarrow \\ A(X') & \xrightarrow{f^*} & A(X) \end{array}$$

commutes. Here, $*$ denotes the contravariant structure while $*$ denotes the covariant structure. The Mackey functor A is *cohomological* if we

further have

$$f_*f^* = \deg(f)$$

for any $f : X' \rightarrow X$, with X connected. We denote by **Mack** the abelian category of Mackey functors, and by **Mack**_c its full subcategory of cohomological Mackey functors.

2.2. Classically [30, (1.4)], a Mackey functor may be viewed as a contravariant additive functor on the category **Span** of “spans” on étale k -schemes, defined as follows: objects are étale k -schemes. A morphism from X to Y is an equivalence class of diagram (span)

$$(2.1) \quad X \xleftarrow{g} Z \xrightarrow{f} Y$$

where, as usual, two spans (Z, f, g) and (Z', f', g') are equivalent if there exists an isomorphism $Z \xrightarrow{\sim} Z'$ making the obvious diagram commute. Composition of spans is defined via fibre product in an obvious manner (compare Quillen’s Q-construction in [18, §2]).

If A is a Mackey functor, the corresponding functor on **Span** has the same value on objects, while its value on a span (2.1) is given by g_*f^* .

Note that **Span** is a preadditive category: one may add (but not subtract) two morphisms with same source and target. We may as well view a Mackey functor as a contravariant additive functor on the associated additive category **Z Span**. In the notation of the appendix, we thus have **Mack** = Mod – **Z Span**.

2.3. Let **Cor** be Voevodsky’s category of finite correspondences on smooth k -schemes, denoted by $SmCor(k)$ in [32, §2.1]. Recall that the objects of **Cor** are the same as the category Sm/k of smooth varieties over k . Following loc. cit., we denote by $c(X, Y)$ the group of morphisms in **Cor** from X to Y , which is, by definition, the free abelian group on the set of closed integral subschemes of $X \times Y$ which are finite and surjective over some irreducible component of X .

Let **PST** be the category of presheaves with transfers (i.e. contravariant additive functors from **Cor** to abelian groups), denoted by $PreShv(SmCor(k))$ in [32, §3.1]. In the same style as §2.2, we have, by definition, **PST** = Mod – **Cor**.

2.4. The category **Z Span** is isomorphic to the full subcategory of **Cor** consisting of smooth k -schemes of dimension 0 (= étale k -schemes). In particular, any presheaf with transfers in the sense of Voevodsky [32, Def. 3.1.1] restricts to a Mackey functor over k . By [31, Cor. 3.15], the restriction of a *homotopy invariant* presheaf with transfers yields a

cohomological Mackey functor. In other words, we have exact functors

$$(2.2) \quad \rho : \mathbf{PST} \rightarrow \mathbf{Mack}$$

$$(2.3) \quad \rho : \mathbf{HI} \rightarrow \mathbf{Mack}_c,$$

where \mathbf{HI} is the full subcategory of \mathbf{PST} consisting of homotopy invariant presheaves with transfers.

2.5. By definition, the functor (2.2) equals i^* , where i is the inclusion $\mathbf{ZSpan} \rightarrow \mathbf{Cor}$. This inclusion has a left adjoint π_0 (scheme of constants). Both functors i and π_0 are symmetric monoidal: for π_0 , reduce to the case where k is algebraically closed.

2.6. Let \mathcal{A} be an additive symmetric monoidal category. In the appendix, we show that the symmetric monoidal structure of \mathcal{A} extends canonically to the category $\text{Mod } -\mathcal{A}$ of (right) \mathcal{A} -modules (§A.8). Given a symmetric monoidal functor $f : \mathcal{A} \rightarrow \mathcal{B}$, its extension $f_!$ to right modules is symmetric monoidal (§A.12).

2.7. If $\mathcal{A} = \mathbf{Cor}$ in §2.6, we get a tensor structure in \mathbf{PST} : we show in Example A.11 that it agrees with the one defined by Voevodsky in [32, p. 206].

For the reader's convenience, we recall how to compute this tensor product (see A.11 and [32, Sect. 3.2]) For a smooth variety X over k , denote as usual by $L(X)$ the Nisnevich sheaf with transfers represented by X . Let \mathcal{F} and \mathcal{F}' be presheaves with transfers. There are exact sequences of the form

$$\begin{aligned} \bigoplus_j L(Y_j) &\rightarrow \bigoplus_i L(X_i) \rightarrow \mathcal{F} \rightarrow 0, \\ \bigoplus_{j'} L(Y_{j'}) &\rightarrow \bigoplus_{i'} L(X_{i'}) \rightarrow \mathcal{F}' \rightarrow 0. \end{aligned}$$

Then the tensor product $\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{F}'$ of \mathcal{F} and \mathcal{F}' is given by

$$\text{Coker} \left(\bigoplus_{j,i'} L(Y_j \times X_{i'}) \oplus \bigoplus_{i,j'} L(X_i \times Y_{j'}) \rightarrow \bigoplus_{i,j} L(X_i \times X_{i'}) \right).$$

2.8. If $\mathcal{A} = \mathbf{ZSpan}$ in §2.6, we get a tensor structure on \mathbf{Mack} . Another tensor product of Mackey functors \otimes^M was originally defined by L. G. Lewis (unpublished); it was used in [8, §5] and [9]. If either A or B is cohomological, $A \otimes^M B$ is cohomological. In Example A.18, we show that \otimes^M agrees with the tensor structure from §2.6.

For the reader's convenience, we recall the definition of $\overset{M}{\otimes}$. Let A_1, \dots, A_n be Mackey functors. For any étale k -scheme X , we define

$$(A_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} A_n)(X) := [\bigoplus_{Y \rightarrow X} A_1(Y) \otimes \dots \otimes A_n(Y)]/R,$$

where $Y \rightarrow X$ runs through all finite étale morphisms, and R is the subgroup generated by all elements of the form

$$a_1 \otimes \dots \otimes f_*(a_i) \otimes \dots \otimes a_n - f^*(a_1) \otimes \dots \otimes a_i \otimes \dots \otimes f^*(a_n),$$

where $Y_1 \xrightarrow{f} Y_2 \rightarrow Y$ is a tower of étale morphisms, $1 \leq i \leq n$, $a_i \in A_i(Y_1)$ and $a_j \in A_j(Y_2)$ ($j = 1, \dots, i-1, i+1, \dots, n$).

2.9. The functor $\rho = i^* = (\pi_0)_*$ of (2.2) is symmetric monoidal, i.e. if \mathcal{F} and \mathcal{G} are presheaves with transfers, then

$$(2.4) \quad \rho \mathcal{F} \overset{M}{\otimes} \rho \mathcal{G} \xrightarrow{\sim} \rho(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}).$$

Indeed, by (A.1) and the right exactness of $\overset{M}{\otimes}$ and $\otimes_{\mathbf{PST}}$ we reduce to \mathcal{F} and \mathcal{G} representable. But if $L(X) \in \mathbf{PST}$ is the presheaf represented by a smooth k -scheme X , then i^* converts the “atomisation” homomorphism

$$\bigoplus_{x \in X_{(0)}} L(x) \rightarrow L(X)$$

into an isomorphism, and the monoidality of ρ follows. (This also shows the exactness of i_* , which we shall not use here.)

2.10. Let $\mathcal{F} \in \mathbf{PST}$. We define $C_1(\mathcal{F}) \in \mathbf{PST}$ by $C_1(\mathcal{F})(X) = \mathcal{F}(X \times \mathbf{A}^1)$ for all smooth X . For $a \in k = \mathbf{A}^1(k)$, the morphism $X \rightarrow X \times \mathbf{A}^1$, $x \mapsto (x, a)$ defines a morphism $i_a^* : C_1(\mathcal{F}) \rightarrow \mathcal{F}$ in \mathbf{PST} .

The inclusion functor $\mathbf{HI} \rightarrow \mathbf{PST}$ has a left adjoint h_0 given by $h_0(\mathcal{F}) = \text{Coker}(i_0^* - i_1^* : C_1(\mathcal{F}) \rightarrow \mathcal{F})$, and the symmetric monoidal structure of \mathbf{PST} induces one on \mathbf{HI} via h_0 . In other words, if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}$, we define

$$(2.5) \quad \mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G} = h_0(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}).$$

Note that (2.3) is *not* symmetric monoidal (since it is the restriction of (2.2)).

2.11. For any $\mathcal{F} \in \mathbf{PST}$, the unit morphism $\mathcal{F} \rightarrow h_0(\mathcal{F})$ induces a surjection

$$(2.6) \quad \mathcal{F}(k) \twoheadrightarrow h_0(\mathcal{F})(k).$$

This is obvious from the formula $h_0(\mathcal{F}) = \text{Coker}(C_1(\mathcal{F}) \rightarrow \mathcal{F})$.

2.12. We shall also need to work with Nisnevich sheaves with transfers. We denote by **NST** the category of Nisnevich sheaves with transfers (objects of **PST** which are sheaves in the Nisnevich topology). By [32, Theorem 3.1.4], the inclusion functor **NST** \rightarrow **PST** has an exact left adjoint $\mathcal{F} \mapsto \mathcal{F}_{\text{Nis}}$ (sheafification). The category **NST** then inherits a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{NST}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{\text{Nis}}.$$

Similarly, we define $\mathbf{HI}_{\text{Nis}} = \mathbf{HI} \cap \mathbf{NST}$. The sheafification functor restricts to an exact functor $\mathbf{HI} \rightarrow \mathbf{HI}_{\text{Nis}}$ [32, Theorem 3.1.11], and \mathbf{HI}_{Nis} gets a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G})_{\text{Nis}}.$$

To summarize, all functors in the following naturally commutative diagram are symmetric monoidal:

$$(2.7) \quad \begin{array}{ccc} \mathbf{PST} & \xrightarrow{\text{Nis}} & \mathbf{NST} \\ h_0 \downarrow & & h_0^{\text{Nis}} \downarrow \\ \mathbf{HI} & \xrightarrow{\text{Nis}} & \mathbf{HI}_{\text{Nis}}. \end{array}$$

where each functor is left adjoint to the corresponding inclusion.

2.13. Let \mathcal{F} be a presheaf on Sm/k , and let \mathcal{F}_{Nis} be the associated Nisnevich sheaf. Then we have an isomorphism

$$(2.8) \quad \mathcal{F}(k) \xrightarrow{\sim} \mathcal{F}_{\text{Nis}}(k).$$

Indeed, any covering of $\text{Spec } k$ for the Nisnevich topology refines to a trivial covering. In particular, the functor $\mathcal{F} \mapsto \mathcal{F}_{\text{Nis}}(k)$ is exact.

This applies in particular to a presheaf with transfers and the associated Nisnevich sheaf with transfers.

2.14. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. Then (2.4) yields a canonical isomorphism

$$(2.9) \quad (\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k) \simeq (\mathcal{F}_1 \otimes_{\mathbf{PST}} \dots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k).$$

Composing (2.9) with the unit morphism $Id \Rightarrow h_0^{\text{Nis}}$ from (2.7) and taking (2.5) into account, we get a canonical morphism

$$(2.10) \quad (\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k) \rightarrow (\mathcal{F}_1 \otimes_{\mathbf{HI}_{\text{Nis}}} \dots \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{F}_n)(k).$$

which is surjective by §§2.11 and 2.13.

2.15. If G is a commutative k -group scheme whose identity component is a quasi-projective variety, then G has a canonical structure of Nisnevich sheaf with transfers ([25, proof of Lemma 3.2] completed by [2, Lemma 1.3.2]). This applies in particular to semi-abelian varieties and also to the "full" Albanese scheme [19] of a smooth variety (which is an extension of a lattice by a semi-abelian variety). In particular, if G_1, \dots, G_n are such k -group schemes, (2.10) yields a canonical surjection

$$(2.11) \quad (G_1 \otimes^M \dots \otimes^M G_n)(k) \rightarrow (G_1 \otimes_{\mathbf{HI}_{\text{Nis}}} \dots \otimes_{\mathbf{HI}_{\text{Nis}}} G_n)(k),$$

where the G_i are considered on the left as Mackey functors, and on the right as homotopy invariant Nisnevich sheaves with transfers.

3. PRESHEAVES WITH TRANSFERS AND MOTIVES

3.1. The left adjoint h_0^{Nis} in (2.7) "extends" to a left adjoint C_* of the inclusion

$$\mathbf{DM}_-^{\text{eff}} \rightarrow D^-(\mathbf{NST})$$

where the left hand side is Voevodsky's triangulated category of effective motivic complexes [32, §3, esp. Prop. 3.2.3].

More precisely, $\mathbf{DM}_-^{\text{eff}}$ is defined as the full subcategory of objects of $D^-(\mathbf{NST})$ whose cohomology sheaves are homotopy invariant. The canonical t -structure of $D^-(\mathbf{NST})$ induces a t -structure on $\mathbf{DM}_-^{\text{eff}}$, with heart \mathbf{HI}_{Nis} . The functor C_* is right exact with respect to these t -structures, and if $\mathcal{F} \in \mathbf{NST}$, then $H_0(C_*(\mathcal{F})) = h_0^{\text{Nis}}(\mathcal{F})$.

3.2. The tensor structure of §2.12 on \mathbf{NST} extends to one on $D^-(\mathbf{NST})$ [32, p. 206]. Via C_* , this tensor structure descends to a tensor structure on $\mathbf{DM}_-^{\text{eff}}$ [32, p. 210], which will simply be denoted by \otimes . The relationship between this tensor structure and the one of §2.12 is as follows: if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\text{Nis}}$, then

$$(3.1) \quad \mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G} = H^0(\mathcal{F}[0] \otimes \mathcal{G}[0])$$

where $\mathcal{F}[0], \mathcal{G}[0]$ are viewed as complexes of Nisnevich sheaves with transfers concentrated in degree 0.

We shall need the following lemma, which is not explicit in [32]:

3.3. Lemma. *The tensor product \otimes of $\mathbf{DM}_-^{\text{eff}}$ is right exact with respect to the homotopy t -structure.*

Proof. By definition,

$$C \otimes D = C_*(C \overset{L}{\otimes} D)$$

for $C, D \in \mathbf{DM}_-^{\text{eff}}$, where $\overset{L}{\otimes}$ is the tensor product of $D^-(\mathbf{NST})$ defined in [32, p. 206]. We want to show that, if C and D are concentrated in degrees ≤ 0 , then so is $C \otimes D$. Using the canonical left resolutions of loc. cit., it is enough to do it for C and D of the form $C_*(L(X))$ and $C_*(L(Y))$ for two smooth schemes X, Y . Since C_* is symmetric monoidal, we have

$$C_*(L(X)) \otimes C_*(L(Y)) \xleftarrow{\sim} C_*(L(X) \overset{L}{\otimes} L(Y)) = C_*(L(X \times Y))$$

and the claim is obvious in view of the formula for C_* [32, p. 207]. \square

3.4. Let $C \in \mathbf{DM}_-^{\text{eff}}$. For any $X \in Sm/k$ and any $i \in \mathbf{Z}$, we have

$$\mathbb{H}_{\text{Nis}}^i(X, C) \simeq \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(M(X), C[i])$$

where $M(X) = C_*(L(X))$ is the motive of X computed in $\mathbf{DM}_-^{\text{eff}}$ (cf. [32, Prop. 3.2.7]).

Specializing to the case $X = \text{Spec } k$ ($M(X) = \mathbf{Z}$) and taking §2.13 into account, we get

$$(3.2) \quad \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathbf{Z}, C[i]) \simeq H^i(C)(k).$$

Combining (3.1), (2.8) and (3.2), we get:

3.5. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then we have a canonical isomorphism*

$$(3.3) \quad (\mathcal{F}_1 \otimes_{\mathbf{HI}_{\text{Nis}}} \cdots \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{F}_n)(k) \simeq \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]).$$

3.6. Summarizing, for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$ we get a surjective homomorphism

$$(3.4) \quad (\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \rightarrow \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]).$$

by composing (2.9), (2.6), (2.5), (2.8) and (3.3).

4. PRESHEAVES WITH TRANSFERS AND LOCAL SYMBOLS

4.1. Given a presheaf with transfers \mathcal{G} , recall from [31, p. 96] the presheaf with transfers \mathcal{G}_{-1} defined by the formula

$$(4.1) \quad \mathcal{G}_{-1}(U) = \text{Coker}(\mathcal{G}(U \times \mathbf{A}^1) \rightarrow \mathcal{G}(U \times (\mathbf{A}^1 - \{0\}))).$$

If $\mathcal{G} \in \mathbf{HI}_{\text{Nis}}$, then $\mathcal{G}(U \times (\mathbf{A}^1 - \{0\})) \simeq \mathcal{G}(U) \oplus \mathcal{G}_{-1}(U)$ for all smooth U . Thus $\mathcal{G}_{-1} \in \mathbf{HI}_{\text{Nis}}$ and $\mathcal{G} \mapsto \mathcal{G}_{-1}$ is an exact endofunctor of \mathbf{HI}_{Nis} .

Suppose that $\mathcal{G} \in \mathbf{HI}_{\text{Nis}}$. Let $X \in Sm/k$ (connected), $K = k(X)$ and $x \in X$ be a point of codimension 1. By [31, Lemma 4.36], there is a canonical isomorphism

$$(4.2) \quad \mathcal{G}_{-1}(k(x)) \simeq H_{\text{Zar}, x}^1(X, \mathcal{G})$$

yielding a canonical map

$$(4.3) \quad \partial_x : \mathcal{G}(K) \rightarrow \mathcal{G}_{-1}(k(x)).$$

The following lemma follows from the construction of the isomorphisms (4.2). It is part of the general fact that \mathcal{G} defines a cycle module in the sense of Rost (cf. [5, Prop. 5.4.64]).

4.2. Lemma. *a) Let $f : Y \rightarrow X$ be a dominant morphism, with Y smooth and connected. Let $L = k(Y)$, and let $y \in Y^{(1)}$ be such that $f(y) = x$. Then the diagram*

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \mathcal{G}_{-1}(k(y)) \\ f^* \uparrow & & e f^* \uparrow \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

commutes, where e is the ramification index of v_y relative to v_x .

b) If f is finite surjective, the diagram

$$\begin{array}{ccc} \mathcal{G}(L) & \xrightarrow{(\partial_y)} & \bigoplus_{y \in f^{-1}(x)} \mathcal{G}_{-1}(k(y)) \\ f_* \downarrow & & f_* \downarrow \\ \mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x)) \end{array}$$

commutes. □

4.3. Proposition. *Let $\mathcal{G} \in \mathbf{HI}_{\text{Nis}}$. There is a canonical isomorphism*

$$\mathcal{G}_{-1} = \underline{\text{Hom}}(\mathbf{G}_m, \mathcal{G}).$$

Proof. The statement means that \mathcal{G}_{-1} represents the functor

$$\mathcal{H} \mapsto \text{Hom}_{\mathbf{HI}_{\text{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m, \mathcal{G}).$$

Sheafifying (4.1) for the Nisnevich topology and using homotopy invariance, we obtain an isomorphism

$$\text{Coker}(\mathcal{G} \rightarrow p_* p^* \mathcal{G}) \xrightarrow{\sim} \mathcal{G}_{-1}$$

where $p : \mathbf{A}^1 - \{0\} \rightarrow \text{Spec } k$ is the structural morphism. Moreover, [31, Prop. 5.4] shows $R^i p_* p^* \mathcal{G}(K) = H_{\text{Nis}}^i(\mathbf{A}_K^1 - \{0\}, p^* \mathcal{G}) = 0$ for any field K/k and $i > 0$, hence by [31, Prop. 4.20] one has $R^i p_* p^* \mathcal{G} = 0$ for any $i > 0$. It follows that $p_* p^* \mathcal{G}[0] \xrightarrow{\sim} R p_* p^* \mathcal{G}[0]$.

By [32, Prop. 3.2.8], we have

$$R p_* p^* \mathcal{G}[0] = \underline{\text{Hom}}(M(\mathbf{A}^1 - \{0\}), \mathcal{G}[0])$$

where $\underline{\text{Hom}}$ is the (partially defined) internal Hom of $\mathbf{DM}_-^{\text{eff}}$. By [32, Prop. 3.5.4] (Gysin triangle) and homotopy invariance, we have an exact triangle, split by any rational point of $\mathbf{A}^1 - \{0\}$:

$$\mathbf{Z}(1)[1] \rightarrow M(\mathbf{A}^1 - \{0\}) \rightarrow \mathbf{Z} \xrightarrow{+1}$$

To get a canonical splitting, we may choose the rational point $1 \in \mathbf{A}^1 - \{0\}$.

By [32, Cor. 3.4.3], we have an isomorphism $\mathbf{Z}(1)[1] \simeq \mathbf{G}_m[0]$. Hence, in $\mathbf{DM}_-^{\text{eff}}$, we have an isomorphism

$$(4.4) \quad \mathcal{G}_{-1}[0] \simeq \underline{\text{Hom}}_{\mathbf{DM}_-^{\text{eff}}}(\mathbf{G}_m[0], \mathcal{G}[0]).$$

Let $\mathcal{H} \in \mathbf{HI}_{\text{Nis}}$. We get:

$$\begin{aligned} & \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathcal{H}[0], \mathcal{G}_{-1}[0]) \\ & \simeq \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathcal{H}[0] \otimes \mathbf{G}_m[0], \mathcal{G}[0]) && \text{by (4.4)} \\ & \simeq \text{Hom}_{\mathbf{HI}_{\text{Nis}}}(H^0(\mathcal{H}[0] \otimes \mathbf{G}_m[0]), \mathcal{G}) && \text{by Lemma 3.3} \\ & = \text{Hom}_{\mathbf{HI}_{\text{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m, \mathcal{G}) && \text{by (3.1)} \end{aligned}$$

as desired. \square

4.4. *Remark.* The proof of Proposition 4.3 also shows that, in $\mathbf{DM}_-^{\text{eff}}$, we have an isomorphism

$$\underline{\text{Hom}}(\mathbf{G}_m[0], \mathcal{G}[0]) \simeq \underline{\text{Hom}}(\mathbf{G}_m, \mathcal{G})[0]$$

where the left $\underline{\text{Hom}}$ is computed in $\mathbf{DM}_-^{\text{eff}}$ and the right $\underline{\text{Hom}}$ is computed in \mathbf{HI}_{Nis} . In particular, $\underline{\text{Hom}}(\mathbf{G}_m[0], -) : \mathbf{DM}_-^{\text{eff}} \rightarrow \mathbf{DM}_-^{\text{eff}}$ is t -exact.

4.5. **Proposition.** *Let C be a smooth, proper, connected curve over k , with function field K . For any $\mathcal{G} \in \mathbf{HI}_{\text{Nis}}$, there exists a canonical homomorphism*

$$\text{Tr}_{C/k} : H_{\text{Zar}}^1(C, \mathcal{G}) \rightarrow \mathcal{G}_{-1}(k)$$

such that, for any $x \in C$, the composition

$$\mathcal{G}_{-1}(k(x)) \simeq H_x^1(C, \mathcal{G}) \rightarrow H_{\text{Zar}}^1(C, \mathcal{G}) \xrightarrow{\text{Tr}_C} \mathcal{G}_{-1}(k)$$

equals the transfer map $\text{Tr}_{k(x)/k}$ associated to the finite surjective morphism $\text{Spec } k(x) \rightarrow \text{Spec } k$.

Proof. By [32, Prop. 3.2.7], we have

$$H_{\text{Zar}}^1(C, \mathcal{G}) \xrightarrow{\sim} H_{\text{Nis}}^1(C, \mathcal{G}) \simeq \text{Hom}_{\mathbf{DM}_-^{\text{eff}}}(M(C), \mathcal{G}[1]).$$

The structural morphism $C \rightarrow \text{Spec } k$ yields a morphism of motives $M(C) \rightarrow \mathbf{Z}$ which, by Poincaré duality [32, Th. 4.3.2], yields a canonical morphism

$$\mathbf{G}_m[1] \simeq \mathbf{Z}(1)[2] \rightarrow M(C).$$

(One may view this morphism as the image of the canonical morphism $\mathbb{L} \rightarrow h(C)$ in the category of Chow motives.)

Therefore, by Proposition 4.3 and Remark 4.4, we get a map

$$\text{Tr}_{C/k} : H_{\text{Zar}}^1(X, \mathcal{G}) \rightarrow \text{Hom}_{\text{DM}_{-}^{\text{eff}}}(\mathbf{G}_m[1], \mathcal{G}[1]) = \mathcal{G}_{-1}(k).$$

It remains to prove the claimed compatibility. Let $M^x(C)$ be the motive of C with supports in x , defined as $C_*(\text{Coker}(L(C - \{x\}) \rightarrow L(C)))$. Let $\mathbf{Z}_{k(x)} = M(\text{Spec } k(x))$. By [32, proof of Prop. 3.5.4], we have an isomorphism $M^x(C) \simeq \mathbf{Z}_{k(x)}(1)[2]$, and we have to show that the composition

$$\mathbf{Z}(1)[2] \rightarrow M(C) \xrightarrow{g_x} \mathbf{Z}_{k(x)}(1)[2]$$

induces $\text{Tr}_{k(x)/k}$, up to twisting and shifting. To see this, we observe that g_x is the image of the morphism of Chow motives

$$h(C) \rightarrow h(\text{Spec } k(x))(1)$$

dual to the morphism $h(\text{Spec } k(x)) \rightarrow h(C)$ induced by the inclusion $\text{Spec } k(x) \rightarrow C$: this is easy to check from the definition of g_x in [32] (observe that in this special case, $\text{Bl}_x(C) = C$ and that we may use a variant of the said construction replacing $C \times \mathbf{A}^1$ by $C \times \mathbf{P}^1$ to stay within smooth projective varieties). The conclusion now follows from the fact that the composition

$$\text{Spec } k(x) \rightarrow C \rightarrow \text{Spec } k$$

is the structural morphism of $\text{Spec } k(x)$. \square

4.6. Proposition (Reciprocity). *Let C be a smooth, proper, connected curve over k , with function field K . Then the sequence*

$$\mathcal{G}(K) \xrightarrow{(\partial_x)} \bigoplus_{x \in C} \mathcal{G}_{-1}(k(x)) \xrightarrow{\sum_x \text{Tr}_{k(x)/k}} \mathcal{G}_{-1}(k)$$

is a complex.

Proof. This follows from Proposition 4.5, since the composition

$$\mathcal{G}(K) \rightarrow \bigoplus_{x \in C} H_x^1(C, \mathcal{G}) \xrightarrow{(g_x)} H^1(C, \mathcal{G})$$

is 0. \square

4.7. Theorem. *Suppose $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$. Then there exists a canonical isomorphism*

$$\mathcal{F} \simeq (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m)_{-1}.$$

Proof. We compute again in $\mathbf{DM}_{-}^{\text{eff}}$. As recalled in the proof of Proposition 4.3, we have $\mathbf{G}_m[0] = \mathbf{Z}(1)[1]$. Hence the cancellation theorem [34] yields a canonical isomorphism

$$\mathcal{F}[0] \simeq \underline{\text{Hom}}_{\mathbf{DM}_{-}^{\text{eff}}}(\mathbf{G}_m[0], \mathcal{F}[0] \otimes \mathbf{G}_m[0])$$

By taking H^0 , we obtain

$$\mathcal{F} \simeq \underline{\text{Hom}}_{\mathbf{HI}_{\text{Nis}}}(\mathbf{G}_m, \mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m).$$

The right hand side is isomorphic to $(\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m)_{-1}$ by Proposition 4.3. \square

4.8. If \mathcal{F}, \mathcal{G} are presheaves with transfers, there is a bilinear morphism of presheaves with transfers (i.e. a natural transformation over $\mathbf{PST} \times \mathbf{PST}$):

$$\begin{aligned} \mathcal{F}(U) \otimes \mathcal{G}_{-1}(V) &= \\ \text{Coker}(\mathcal{F}(U) \otimes \mathcal{G}(V \times \mathbf{A}^1) &\rightarrow \mathcal{F}(U) \otimes \mathcal{G}(V \times (\mathbf{A}^1 - \{0\}))) \rightarrow \\ \text{Coker}((\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})(U \times V \times \mathbf{A}^1) &\rightarrow (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})(U \times V \times (\mathbf{A}^1 - \{0\}))) \\ &= (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}(U \times V) \end{aligned}$$

which induces a morphism

$$(4.5) \quad \mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}_{-1} \rightarrow (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}.$$

4.9. Notation. Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\text{Nis}}$ and $\mathcal{H} = \mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G}$. Let X, K, x be as in §4.1. For $(a, b) \in \mathcal{F}(K) \times \mathcal{G}(K)$, we denote by $a \cdot b$ the image of $a \otimes b$ in $\mathcal{H}(K)$ by the map

$$(4.6) \quad \mathcal{F}(K) \otimes \mathcal{G}(K) \rightarrow \mathcal{H}(K).$$

We define the *local symbol* on \mathcal{F}

$$\mathcal{F}(K) \times K^* \rightarrow \mathcal{F}(k(x))$$

to be the composition

$$\mathcal{F}(K) \times K^* \xrightarrow{\quad} (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m)(K) \xrightarrow{\partial_x} (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathbf{G}_m)_{-1}(k(x)) \simeq \mathcal{F}(k(x))$$

where the first map is given by (4.6) with $\mathcal{G} = \mathbf{G}_m$, and the last isomorphism is given by Theorem 4.7. The image of $(a, b) \in \mathcal{F}(K) \times K^*$ by the local symbol is denoted by $\partial_x(a, b) \in \mathcal{F}(k(x))$.

4.10. **Proposition** (cf. [5, Prop. 5.5.27]). *Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\text{Nis}}$, and consider the morphism induced by (4.5)*

$$\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G}_{-1} \xrightarrow{\Phi} (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G})_{-1}.$$

Let X, K, x be as in §4.1. Then the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathcal{O}_{X,x}) \otimes \mathcal{G}(K) & \longrightarrow & (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G})(K) \\ \downarrow i_x^* \otimes \partial_x & & \downarrow \partial_x \\ \mathcal{F}(k(x)) \otimes \mathcal{G}_{-1}(k(x)) & & \\ \downarrow & & \downarrow \Phi \\ (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G}_{-1})(k(x)) & \xrightarrow{\Phi} & (\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G})_{-1}(k(x)) \end{array}$$

commutes, where i_x^* is induced by the reduction map $\mathcal{O}_{X,x} \rightarrow k(x)$. In other words, with Notation 4.9 we have the identity

$$(4.7) \quad \partial_x(a \cdot b) = \Phi(i_x^* a \cdot \partial_x b)$$

for $(a, b) \in \mathcal{F}(\mathcal{O}_{X,x}) \times \mathcal{G}(K)$.

4.11. **Corollary.** *Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$; let X, K, x be as in §4.1 and let $(a, f) \in \mathcal{F}(K) \times K^*$.*

a) *Suppose that there is $\tilde{a} \in \mathcal{F}(\mathcal{O}_{X,x})$ whose image in $\mathcal{F}(K)$ is a .³ Then we have*

$$\partial_x(a, f) = v_x(f)a(x)$$

where $a(x)$ is the image of \tilde{a} in $\mathcal{F}(k(x))$.

b) *Suppose that $v_x(f - 1) > 0$. Then $\partial_x(a, f) = 0$.*

Proof. a) Since (4.3) is given by v_x when $\mathcal{G} = \mathbf{G}_m$, this follows from Proposition 4.10 (applied with $\mathcal{G} = \mathbf{G}_m$) and Theorem 4.7. b) This follows again from Proposition 4.10, after switching the rôles of \mathcal{F} and \mathcal{G} . \square

4.12. **Proposition.** *Let G be a semi-abelian variety. The local symbol on G defined in Notation 4.9 agrees with Somekawa's local symbol [24, (1.1)] (generalising the Rosenlicht-Serre local symbol) on G .*

Proof. Up to base-changing from k to \bar{k} (see Lemma 4.2 a)), we may assume k algebraically closed. By [23, Ch. III, Prop. 1], it suffices to show that the local symbol in Notation 4.9 satisfies the properties in [23, Ch. III, Def. 2] which characterize the Rosenlicht-Serre local symbol. In this definition, Condition i) is obvious, Condition ii) is Corollary 4.11 b), Condition iii) is Corollary 4.11 a) and Condition iv) is Proposition 4.6. \square

³By [31, Cor. 4.19], \tilde{a} is unique.

5. K -GROUPS OF SOMEKAWA TYPE

5.1. Definition. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$.

a) A *relation datum of Somekawa type* for $\mathcal{F}_1, \dots, \mathcal{F}_n$ is a collection $(C, h, (g_i)_{i=1, \dots, n})$ of the following objects: (i) a smooth proper connected curve C over k , (ii) $h \in k(C)^*$, and (iii) $g_i \in \mathcal{F}_i(k(C))$ for each $i \in \{1, \dots, n\}$; which satisfies the condition

$$(5.1) \quad \text{for any } c \in C, \text{ there is } i(c) \text{ such that } c \in R_i \text{ for all } i \neq i(c),$$

where $R_i := \{c \in C \mid g_i \in \text{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \rightarrow \mathcal{F}_i(k(C)))\}$.

b) We define the K -group of Somekawa type $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ to be the quotient of $(\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$(5.2) \quad \sum_{c \in C} \text{Tr}_{k(c)/k}(g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)}, h) \otimes \dots \otimes g_n(c))$$

where $(C, h, (g_i)_{i=1, \dots, n})$ runs through all relation data of Somekawa type.

5.2. Remark. In view of Proposition 4.12, our group $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ coincides with the Milnor K -group defined in [24] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semi-abelian varieties over k .⁴ (Note that Somekawa works with all finite extensions but over an arbitrary field; we work with finite separable extensions but are assuming k perfect.)

5.3. Theorem. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. The homomorphism (2.10) factors through $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we get a surjective homomorphism (1.1).

Proof. Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{HI}_{\text{Nis}}} \dots \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{F}_n$. Let $(C, h, (g_i)_{i=1, \dots, n})$ be a relation datum of Somekawa type. We must show that the element (5.2) goes to 0 in $\mathcal{F}(k)$ via (2.10). Consider the element $g = g_1 \otimes \dots \otimes g_n \in \mathcal{F}(K)$. It follows from (4.7) that, for any $c \in C$, we have

$$\begin{aligned} & g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)}, h) \otimes \dots \otimes g_n(c) \\ &= g_1(c) \otimes \dots \otimes \partial_c(g_{i(c)} \otimes \{h\}) \otimes \dots \otimes g_n(c) = \partial_c(g \otimes \{h\}). \end{aligned}$$

The claim now follows from Proposition 4.6. \square

⁴As was observed by W. Raskind, the signs appearing in [24, (1.2.2)] should not be there (cf. [20, p. 10, footnote]).

6. K -GROUPS OF GEOMETRIC TYPE

6.1. **Definition.** Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$.

a) A *relation datum of geometric type* for $\mathcal{F}_1, \dots, \mathcal{F}_n$ is a collection $(C, f, (g_i)_{i=1, \dots, n})$ of the following objects: (i) a smooth projective connected curve C over k , (ii) a surjective morphism $f : C \rightarrow \mathbf{P}^1$, (iii) $g_i \in \mathcal{F}_i(C')$ for each $i \in \{1, \dots, n\}$, where $C' = f^{-1}(\mathbf{P}^1 \setminus \{1\})$.

b) We define the K -group of geometric type $K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ to be the quotient of $(\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$(6.1) \quad \sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k}(g_1(c) \otimes \dots \otimes g_n(c))$$

where $(C, f, (g_i)_{i=1, \dots, n})$ runs through all relation data of geometric type. (Here we used the notation $g_i(c) = \iota_c^*(g_i) \in \mathcal{F}(k(c))$, where $\iota_c : c = \operatorname{Spec} k(c) \rightarrow C'$ is the closed immersion.)

The rest of this section is devoted to a proof of the following theorem:

6.2. **Theorem.** Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. The homomorphism (2.10) induces an isomorphism

$$(6.2) \quad K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{DM}_{\text{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \dots \otimes \mathcal{F}_n[0]).$$

6.3. *Remark.* By combining Theorems 5.3 and 6.2, we obtain a surjective homomorphism

$$(6.3) \quad K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \rightarrow K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$$

for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. The existence of this surjection is *not* clear from the definition.

6.4. Let $\mathcal{F} \in \mathbf{PST}$. Suppose that we are given the following data: (i) a smooth projective connected curve C over k , (ii) a surjective morphism $f : C \rightarrow \mathbf{P}^1$, (iii) a map $\alpha : L(C') \rightarrow \mathcal{F}$ in \mathbf{PST} , where $C' = f^{-1}(\Delta)$ and $\Delta = \mathbf{P}^1 \setminus \{1\} (\simeq \mathbf{A}^1)$. To such a triple (C, f, α) , we associate an element

$$(6.4) \quad \alpha(\operatorname{div}(f)) \in \mathcal{F}(k),$$

where we regard $\operatorname{div}(f)$ as an element of $Z_0(C') = c(\operatorname{Spec} k, C') = L(C')(k)$.

One can rewrite the element (6.4) as follows. The map $\alpha : L(C') \rightarrow \mathcal{F}$ can be regarded as a section $\alpha \in \mathcal{F}(C')$. To each closed point $c \in C'$, we write $\alpha(c)$ for the image of α in $\mathcal{F}(k(c))$ by the map induced by $c = \operatorname{Spec} k(c) \rightarrow C'$. Now (6.4) is rewritten as

$$(6.5) \quad \sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k} \alpha(c).$$

6.5. Proposition. *Let $\mathcal{F} \in \mathbf{PST}$. We define $\mathcal{F}(k)_{\text{rat}}$ to be the subgroup of $\mathcal{F}(k)$ generated by elements (6.4) for all triples (C, f, α) as in §6.4. Then we have*

$$h_0(\mathcal{F})(k) = \mathcal{F}(k)/\mathcal{F}(k)_{\text{rat}}.$$

Proof. By definition we have

$$(6.6) \quad h_0(\mathcal{F})(k) = \text{Coker}(i_0^* - i_\infty^* : \mathcal{F}(\Delta) \rightarrow \mathcal{F}(k)),$$

where $\Delta = \mathbf{P}^1 \setminus \{1\} (\simeq \mathbf{A}^1)$ and i_a^* is the pull-back by the inclusion $i_a : \{a\} \rightarrow \Delta$ for $a \in \{0, \infty\}$.

Suppose we are given a triple (C, f, α) as in §6.4, and set $C' = f^{-1}(\Delta)$. The graph $\gamma_{f|_{C'}}$ of $f|_{C'}$ defines an element of $c(\Delta, C') = L(C')(\Delta)$. In the commutative diagram

$$\begin{array}{ccc} L(C')(\Delta) & \xrightarrow{\alpha} & \mathcal{F}(\Delta) \\ i_0^* - i_\infty^* \downarrow & & \downarrow i_0^* - i_\infty^* \\ L(C')(k) & \xrightarrow{\alpha} & \mathcal{F}(k), \end{array}$$

the image of $\gamma_{f|_{C'}}$ in $L(C')(k) = Z_0(C')$ is $\text{div}(f)$, which shows the vanishing of $\alpha(\text{div}(f))$ in $h_0(\mathcal{F})(k)$.

Conversely, given $\alpha \in \mathcal{F}(\Delta)$, (6.4) for the triple $(\mathbf{P}^1, \text{id}_{\mathbf{P}^1}, \alpha)$ coincides with $(i_0^* - i_\infty^*)(\alpha)$. This completes the proof. \square

6.6. Lemma. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{PST}} \dots \otimes_{\mathbf{PST}} \mathcal{F}_n$. Let (C, f, α) be a triple considered in §6.4. Then $\alpha \in \mathcal{F}(C')$ is the sum of a finite number of elements of the form*

$$(6.7) \quad \text{Tr}_h(g_1 \otimes \dots \otimes g_n),$$

where D is a smooth projective curve, $h : D \rightarrow C$ is a surjective morphism, $g_i \in \mathcal{F}_i(h^{-1}(C'))$ for $i = 1, \dots, n$, and $\text{Tr}_h : \mathcal{F}(h^{-1}(C')) \rightarrow \mathcal{F}(C')$ is the transfer with respect to $h|_{h^{-1}(C')}$.

Proof. By the facts recalled in §2.7, we are reduced to the case $\mathcal{F}_i = L(X_i)$ where X_i is a smooth variety over k for each $i = 1, \dots, n$. Then we have $\mathcal{F} = L(X)$ with $X = X_1 \times \dots \times X_n$. Let Z be an integral closed subscheme of $C' \times X$ which is finite and surjective over C' . It suffices to show that $Z \in c(C', X) = L(X)(C')$ can be written as (6.7).

Let $q : D' \rightarrow Z$ be the normalization, and let $h : D' \rightarrow C'$ be the composition $D' \rightarrow Z \rightarrow C'$, so that h is a finite surjective morphism. For $i = 1, \dots, n$, we define $g_i \in c(D', X_i) = L(X_i)(D')$ to be the graph of $D' \rightarrow X \rightarrow X_i$. If we set $g = g_1 \otimes \dots \otimes g_n \in L(X)(D')$, then by definition we have $\text{Tr}_h(g) = Z$ in $L(X)(C')$. The assertion is proved. \square

6.7. Now it follows from Definition 6.1 b), Proposition 6.5, Lemma 6.6 and (6.5) that (2.9) and (2.6) induce an isomorphism

$$K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \simeq h_0(\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$$

for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. If $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$, the right hand side is canonically isomorphic to $\text{Hom}_{\mathbf{DM}_{\text{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0])$ by (3.3) and (2.8). This completes the proof of Theorem 6.2. \square

7. MILNOR K -THEORY

7.1. Our aim is to show that the map (6.3) is bijective. The first step is the special case of the multiplicative groups.

Recall that, if C is a smooth projective connected curve over k , then the composition

$$K_{r+1}^M(k(C)) \xrightarrow{\partial_c} \bigoplus_{c \in C} K_r^M(k(c)) \xrightarrow{\oplus N_{k(c)/k}} K_r^M(k)$$

is the zero map by Weil reciprocity [3, Ch. I, (5.4)]. Here, for each closed point $c \in C$, we write $\partial_c : K_{r+1}^M(k(C)) \rightarrow K_r^M(k(c))$ and $N_{k(c)/k} : K_r^M(k(c)) \rightarrow K_r^M(k)$ for the tame symbol and the norm map. The tame symbol satisfies (and is characterized by) the property

$$\partial_c(\{a_1, \dots, a_n, f\}) = v_c(f)\{a_1(c), \dots, a_n(c)\}$$

for any $a_1, \dots, a_n \in \mathcal{O}_{C,c}^*$ and $f \in k(C)^*$.

7.2. Proposition. *When $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbf{G}_m$, the map (6.3) is bijective.*

Proof. It suffices to show that relations (6.1) vanish in $K(k; \mathbf{G}_m, \dots, \mathbf{G}_m)$. Because of Somekawa's isomorphism [24, Theorem 1.4]

$$(7.1) \quad K(k; \mathbf{G}_m, \dots, \mathbf{G}_m) \simeq K_n^M(k)$$

given by $\{x_1, \dots, x_n\}_{E/k} \mapsto N_{E/k}(\{x_1, \dots, x_n\})$, it suffices to show this vanishing in the usual Milnor K -group $K_n^M(k)$, which follows from Weil reciprocity recalled above. \square

The following lemmas appear to be crucial in the proof of the main theorem.

7.3. Lemma. *Let C be a smooth projective connected curve over k , and let $Z = \{p_1, \dots, p_s\}$ be a finite set of closed points of C . If k is infinite, then we have $K_2^M(k(C)) = \{k(C)^*, \mathcal{O}_{C,Z}^*\}$.*

Proof. Let \mathfrak{p}_i be the maximal ideal of $A = \mathcal{O}_{C,Z}$ corresponding to p_i . Since A is a semi-local PID, we can choose generators π_1, \dots, π_s of $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Since k is infinite, we can change π_i into $\mu_i \pi_i$ for suitable $\mu_1, \dots, \mu_s \in k^*$ to achieve $\pi_i + \pi_j \not\equiv 0 \pmod{\mathfrak{p}_k}$ for i, j, k all distinct (indeed, the set of bad (μ_1, \dots, μ_s) is contained in a finite union of hyperplanes in \bar{k}^s). It follows that $\pi_i + \pi_j \in A^*$ for all $i \neq j$.

By the relation $\{f, -f\} = 0$ ($f \in k(C)^*$), we have $K_2^M(k(C)) = \{A^*, A^*\} + \sum_{i < j} \{\pi_i, \pi_j\}$. Now the identity

$$\begin{aligned} \{\pi_i, \pi_j\} &= \{\pi_i, \pi_j\} - \{-\pi_j, \pi_j\} = \{-\pi_i/\pi_j, \pi_j\} \\ &= \{-\pi_i/\pi_j, \pi_j\} + \{-\pi_i/\pi_j, 1 + (\pi_i/\pi_j)\} = \{-\pi_i/\pi_j, \pi_i + \pi_j\} \end{aligned}$$

proves the lemma. \square

7.4. Lemma. *Let C be a smooth projective connected curve over k , $Z \subset C$ a proper closed subset, and $r > 0$. If k is an infinite field, then $K_{r+1}^M k(C)$ is generated by elements of the form $\{a_1, \dots, a_{r+1}\}$ where the $a_i \in k(C)^*$ satisfy $\text{Supp}(\text{div}(a_i)) \cap Z = \emptyset$ for all $1 \leq i \leq r$ and $\text{Supp}(\text{div}(a_i)) \cap \text{Supp}(\text{div}(a_j)) = \emptyset$ for all $1 \leq i < j \leq r$.*

Proof. We proceed by induction on r . The assertion follows from Lemma 7.3 when $r = 1$. Suppose $r > 1$. Take $a_1, \dots, a_{r+1} \in k(C)^*$. By induction, there exist $b_{m,i} \in k(C)^*$ such that $\text{Supp}(\text{div}(b_{m,i})) \cap Z = \emptyset$ for all $i < r$ and m , $\text{Supp}(\text{div}(b_{m,i})) \cap \text{Supp}(\text{div}(b_{m,j})) = \emptyset$ for all $i < j < r$ and m , and

$$\{a_1, \dots, a_r\} = \sum_m \{b_{m,1}, \dots, b_{m,r}\}$$

holds in $K_r^M k(C)$. For each m , Lemma 7.3 shows that there exist $c_{m,i}, d_{m,i} \in k(C)^*$ such that

$$\text{Supp}(\text{div}(c_{m,i})) \cap \left(Z \cup \bigcup_{j=1}^{r-1} \text{Supp}(\text{div}(b_{m,j})) \right) = \emptyset$$

and that

$$\{b_{m,r}, a_{r+1}\} = \sum_i \{c_{m,i}, d_{m,i}\}$$

holds in $K_2^M k(C)$. Then we have

$$\{a_1, \dots, a_{r+1}\} = \sum_{m,i} \{b_{m,1}, \dots, b_{m,r-1}, c_{m,i}, d_{m,i}\}$$

in $K_{r+1}^M k(C)$, and we are done. \square

8. K -GROUPS OF MILNOR TYPE

We now generalize the notion of Milnor K -groups to arbitrary homotopy invariant Nisnevich sheaves with transfers, although we shall seriously use this generalization only for special, representable, sheaves.

8.1. Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$. We shall call a homomorphism $\mathbf{G}_m \rightarrow \mathcal{F}$ a *cocharacter* of \mathcal{F} . (By Proposition 4.3, the group $\text{Hom}_{\mathbf{HI}_{\text{Nis}}}(\mathbf{G}_m, \mathcal{F})$ is canonically isomorphic to $\mathcal{F}_{-1}(k)$.)

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. Denote by $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ the subgroup of $(\mathcal{F}_1 \otimes_{\mathbf{PST}} \dots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$ generated by the elements

$$(8.1) \quad a_1 \otimes \dots \otimes \chi_i(a) \otimes \dots \otimes \chi_j(1-a) \otimes \dots \otimes a_n$$

where $\chi_i : \mathbf{G}_m \rightarrow \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \rightarrow \mathcal{F}_j$ are 2 cocharacters with $i < j$, $a \in k^* \setminus \{1\}$, and $a_m \in \mathcal{F}_m(k)$ ($m \neq i, j$).

8.2. **Definition.** For $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$, we write $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ for the quotient of $(\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k)$ by the subgroup generated by $\text{Tr}_{E/k} St(E; \mathcal{F}_1, \dots, \mathcal{F}_n)$, where E runs through all finite extensions of k . This is the *K -group of Milnor type* associated to $\mathcal{F}_1, \dots, \mathcal{F}_n$.

8.3. Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. We have a canonical isomorphism

$$(\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k) \simeq (\mathbf{Z} \otimes^M \dots \otimes^M \mathbf{Z} \otimes^M \mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k)$$

because \mathbf{Z} is the unit object for the tensor structure of Mackey functors. Since there is no non-trivial cocharacter of \mathbf{Z} , it induces an isomorphism

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \simeq \tilde{K}(k; \mathbf{Z}, \dots, \mathbf{Z}, \mathcal{F}_{r+1}, \dots, \mathcal{F}_n).$$

8.4. The assignment $k \mapsto \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ inherits the structure of a cohomological Mackey functor, which is natural in $(\mathcal{F}_1, \dots, \mathcal{F}_n)$. In particular, the choice of elements $f_i \in \mathcal{F}_i(k) = \text{Hom}_{\mathbf{HI}_{\text{Nis}}}(\mathbf{Z}, \mathcal{F}_i)$ for $i = 1, \dots, r$ induces a homomorphism

$$(8.2) \quad \begin{aligned} \tilde{K}(k; \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) &\simeq \tilde{K}(k; \mathbf{Z}, \dots, \mathbf{Z}, \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) \\ &\rightarrow \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n). \end{aligned}$$

8.5. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. The image of $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we have a surjective homomorphism $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \rightarrow K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and a composite surjective homomorphism*

$$(8.3) \quad \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \twoheadrightarrow K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$$

Proof. This is a straightforward generalization of Somekawa's proof of [24, Th. 1.4]. We need to show the image of (8.1) vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. By functoriality, we may assume that $\mathcal{F}_i = \mathcal{F}_j = \mathbf{G}_m$ for some $i < j$ and χ_i, χ_j are the identity cocharacters. Given $a_m \in \mathcal{F}_m(k)$ ($m \neq i, j$) and $a \in k^* \setminus \{1\}$, we put $a_i = 1 - at^{-1}, a_j = 1 - t \in \mathbf{G}_m(k(\mathbf{P}^1)) = k(t)^*$. Then $(\mathbf{P}^1, t, (a_1, \dots, a_n))$ is a relation datum of Somekawa type. Note that $a_i \in \mathbf{G}_m(\mathbf{P}^1 \setminus \{0, a\})$ and $a_j \in \mathbf{G}_m(\mathbf{P}^1 \setminus \{1, \infty\})$. Direct computation shows

$$a_j(0) = \partial_\infty(a_j, t) = \partial_1(a_j, t) = 1, \quad \partial_a(a_i, t) = a^{-1}, \quad a_j(a) = 1 - a.$$

Thus this relation datum yields the vanishing of

$$\{a_1, \dots, a_{i-1}, a^{-1}, a_{i+1}, \dots, a_{j-1}, 1 - a, a_{j+1}, \dots, a_n\}_{k/k},$$

which is the negative of the image of (8.1) in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. \square

8.6. Lemma. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$ and let $\mathcal{G}' \twoheadrightarrow \mathcal{G}''$ be an epimorphism in \mathbf{HI}_{Nis} . If (8.3) is bijective for $(\mathcal{G}', \mathcal{F}_1, \dots, \mathcal{F}_n)$, it is bijective for $(\mathcal{G}'', \mathcal{F}_1, \dots, \mathcal{F}_n)$.*

Proof. Let $\mathcal{G} = \text{Ker}(\mathcal{G}' \rightarrow \mathcal{G}'')$. For $\mathcal{H} \in \{\mathcal{G}, \mathcal{G}', \mathcal{G}''\}$, we put $\tilde{K}_{\mathcal{H}} = \tilde{K}(k; \mathcal{H}, \mathcal{F}_1, \dots, \mathcal{F}_n)$, $K'_{\mathcal{H}} = K'(k; \mathcal{H}, \mathcal{F}_1, \dots, \mathcal{F}_n)$. In the commutative diagram

$$\begin{array}{ccccccc} \tilde{K}_{\mathcal{G}} & \longrightarrow & \tilde{K}_{\mathcal{G}'} & \xrightarrow{f} & \tilde{K}_{\mathcal{G}''} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K'_{\mathcal{G}} & \longrightarrow & K'_{\mathcal{G}'} & \longrightarrow & K'_{\mathcal{G}''} & \longrightarrow & 0 \end{array}$$

the upper row is a complex and f is surjective. The lower row is exact because of Theorem 6.2 and Lemma 3.3, and all vertical arrows are surjective. The assertion now follows from a diagram chase. \square

8.7. Let E/k be an étale k -algebra and let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$. We define the *Weil restriction* $R_{E/k}\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ of \mathcal{F} by the formula $R_{E/k}\mathcal{F}(U) = \mathcal{F}(U \times_k E)$ for all smooth variety U . If \mathcal{F} is a semi-abelian variety, then $R_{E/k}\mathcal{F}$ is the (usual) Weil restriction of \mathcal{F} .

8.8. Lemma. *Let E/k be a finite extension. Let $\mathcal{F}_1, \dots, \mathcal{F}_{n-1} \in \mathbf{HI}_{\text{Nis}}$, and let \mathcal{F}_n be a Nisnevich sheaf with transfers over E . We have canonical isomorphisms*

$$\begin{aligned} K(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) &\simeq K(E; \mathcal{F}_1, \dots, \mathcal{F}_n), \\ K'(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) &\simeq K'(E; \mathcal{F}_1, \dots, \mathcal{F}_n), \\ \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_{n-1}, R_{E/k}\mathcal{F}_n) &\simeq \tilde{K}(E; \mathcal{F}_1, \dots, \mathcal{F}_n). \end{aligned}$$

Proof. The first isomorphism was constructed in [26, Lemma 4] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semi-abelian varieties. The same construction works for arbitrary $\mathcal{F}_1, \dots, \mathcal{F}_n$ and also for K' and \tilde{K} . \square

8.9. If $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathbf{G}_m$, (8.3) is bijective by Proposition 7.2. This is false in general, *e.g.* if all the \mathcal{F}_i are proper (Definition 10.1) and $n > 1$. However, we have:

8.10. **Proposition.** *a) Let $\mathcal{F}_1 = \mathcal{F}'_1 \oplus \mathcal{F}''_1$. Then the natural map*

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \rightarrow \tilde{K}(k; \mathcal{F}'_1, \dots, \mathcal{F}_n) \oplus \tilde{K}(k; \mathcal{F}''_1, \dots, \mathcal{F}_n)$$

is bijective.

b) Let T_1, \dots, T_n be tori. Assume that, for each i , there exists an exact sequence of tori

$$0 \rightarrow P_i^1 \rightarrow P_i^0 \rightarrow T_i \rightarrow 0$$

where P_i^0 and P_i^1 are invertible tori (i.e. direct summands of permutation tori). Then (8.3) is bijective for $\mathcal{F}_i = T_i$.

Proof. a) This is formal, as $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ is a quotient of the multiadditive multifunctor $(\mathcal{F}_1 \otimes^M \dots \otimes^M \mathcal{F}_n)(k)$ (see 8.4).

b) Note that, by Hilbert's theorem 90, the sequences $0 \rightarrow P_i^1 \rightarrow P_i^0 \rightarrow T_i \rightarrow 0$ are exact in $\mathbf{H}\mathbf{I}_{\text{Nis}}$. Lemma 8.6 reduces us to the case where all T_i are permutation tori. Then Lemma 8.8 reduces us to the case where all T_i are split tori. Finally, we reduce to $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathbf{G}_m$ by a). \square

8.11. *Question.* Is proposition 8.10 b) true for general tori?

8.12. Let T_1, \dots, T_n be as in Proposition 8.10 b). Let C/k be a smooth projective connected curve with function field K and $v \in C$ a closed point. Put $\mathcal{F} = T_1 \otimes_{\mathbf{H}\mathbf{I}_{\text{Nis}}} \dots \otimes_{\mathbf{H}\mathbf{I}_{\text{Nis}}} T_n$. By Theorem 4.7 and (4.3), we get a residue map $\partial_v : \mathcal{F} \otimes_{\mathbf{H}\mathbf{I}_{\text{Nis}}} \mathbf{G}_m(K) \rightarrow \mathcal{F}(k(v))$. From Lemma 3.5, Theorem 6.2 and Proposition 8.10 b), this can be reinterpreted as

$$(8.4) \quad \partial_v : \tilde{K}(K; T_1, \dots, T_n, \mathbf{G}_m) \rightarrow \tilde{K}(k(v); T_1, \dots, T_n).$$

As v varies, these maps satisfy the reciprocity law of Proposition 4.6 and the compatibility of Lemma 4.2.

9. REDUCTION TO THE REPRESENTABLE CASE

Following [32, p. 207], we write $h_0^{\text{Nis}}(X) := h_0^{\text{Nis}}(L(X))$ for a smooth variety X over k .

9.1. Proposition. *The following statements are equivalent:*

- a) *The homomorphism (6.3) is bijective for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$.*
- b) *Let $\mathcal{F}_1 = \dots = \mathcal{F}_n = h_0^{\text{Nis}}(C')$ for a smooth connected curve C'/k . Then (6.3) is bijective.*
- c) *Let C be a smooth projective connected curve over k , and let $f : C \rightarrow \mathbf{P}^1$ be a surjective morphism. Let $C' = f^{-1}(\mathbf{P}^1 \setminus \{1\})$. Let $\iota : L(C') \rightarrow h_0^{\text{Nis}}(C') =: \mathcal{A}$ be the canonical surjection, which we regard as an element of $\mathcal{A}(C')$. These data define a relation datum of geometric type $(C, f, (\iota, \dots, \iota))$ for $\mathcal{F}_1 = \dots = \mathcal{F}_n = \mathcal{A}$, and its associated element (6.1) is*

$$(9.1) \quad \sum_{c \in C'} v_c(f) \text{Tr}_{k(c)/k}(\iota(c) \otimes \dots \otimes \iota(c)) \in \mathcal{A}^{\otimes M} \dots \otimes \mathcal{A}(k).$$

Then the image of (9.1) in $K(k; \mathcal{A}, \dots, \mathcal{A})$ vanishes.

Proof. Only the implication c) \Rightarrow a) requires a proof. Let $(C, f, (g_i))$ be a relation datum of geometric type for $\mathcal{F}_1, \dots, \mathcal{F}_n$. We need to show the vanishing of

$$(9.2) \quad \sum_{c \in C'} v_c(f) \{g_1(c), \dots, g_n(c)\}_{k(c)/k} \quad \text{in } K(k; \mathcal{F}_1, \dots, \mathcal{F}_n).$$

For each $i = 1, \dots, n$, the section $g_i : L(C') \rightarrow \mathcal{F}_i$ factors through a morphism $\varphi_i : \mathcal{A} \rightarrow \mathcal{F}_i$ since \mathcal{F}_i is homotopy invariant. The homomorphism $K(k; \mathcal{A}, \dots, \mathcal{A}) \rightarrow K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ defined by $(\varphi_1, \dots, \varphi_n)$ sends the image of (9.1) in $K(k; \mathcal{A}, \dots, \mathcal{A})$ to (9.2). Hence (9.2) vanishes by the assumption c). \square

10. PROPER SHEAVES

10.1. Definition. Let \mathcal{F} be a Nisnevich sheaf with transfers. We call \mathcal{F} *proper* if, for any smooth curve C over k and any closed point $c \in C$, the induced map $\mathcal{F}(\mathcal{O}_{C,c}) \rightarrow \mathcal{F}(k(C))$ is surjective. We say that \mathcal{F} is *universally proper* if the above condition holds when replacing k by any finitely generated extension K/k , and C by any regular K -curve.

10.2. Example. (1) A semi-abelian variety G over k is proper in the sense of Def. 10.1 if and only if G is an abelian variety.

- (2) Recall from [10] that $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ is called *birational* if $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is bijective for any smooth k -variety X and any open dense subset $U \subset X$. A birational sheaf $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ is by definition proper. Examples of birational sheaves will be given in Lemma 11.2 b) below. In particular, if C is a smooth proper curve, then $h_0^{\text{Nis}}(C)$ is proper.

In fact:

10.3. Lemma. *Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$. Then*

a) \mathcal{F} is proper if and only if $\mathcal{F}(C) \xrightarrow{\sim} \mathcal{F}(k(C))$ for any smooth k -curve C .

b) \mathcal{F} is universally proper if and only if it is birational (see Example 10.2 (2)).

Proof. Let us prove b), as the proof of a) is a subset of it. It is obvious from the definitions that birational sheaves are universally proper. Conversely, assume \mathcal{F} to be universally proper. Let X be a smooth k -variety. By [31, Cor. 4.19], the map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is injective for any dense open subset of X . Let $x \in X^{(1)}$ and let $p : X \rightarrow \mathbf{A}^{d-1}$ be a dominant rational map defined at x , where $d = \dim X$. (We may find such a p thanks to Noether's normalization theorem.) Applying the hypothesis to the generic fibre of p , we find that $\mathcal{F}(\mathcal{O}_{X,x}) \rightarrow \mathcal{F}(k(X))$ is surjective. Since this is true for all points $x \in X^{(1)}$, we get the surjectivity of $\mathcal{F}(X) \rightarrow \mathcal{F}(k(X))$ from Voevodsky's Gersten resolution [31, Th. 4.37]. \square

The following proposition is not necessary for the proof of the main theorem, but its proof is much simpler than the general case.

10.4. Proposition. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$. Assume that $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ are proper. Then the homomorphism (6.3) is bijective.*

Proof. Suppose that $(C, f, (g_i)_{i=1, \dots, n})$ is a relation datum of geometric type for $(\mathcal{F}_1, \dots, \mathcal{F}_n)$. It suffices to show the vanishing of the image of

$$(10.1) \quad \sum_{c \in C'} v_c(f) \text{Tr}_{k(c)/k}(g_1(c) \otimes \cdots \otimes g_n(c)) \in (\mathcal{F}_1 \otimes^M \cdots \otimes^M \mathcal{F}_n)(k)$$

in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Let \bar{g}_i be the image of g_i in $\mathcal{F}(k(C))$. By assumption we have $\bar{g}_i \in \text{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \rightarrow \mathcal{F}_i(k(C)))$ for all $c \in C$ and $i = 1, \dots, n-1$. Hence $(C, f, (\bar{g}_i)_{i=1, \dots, n})$ is a relation datum of Somekawa type (with $i(c) = n$ for all $c \in C$). By Corollary 4.11, the element (10.1) coincides with

$$\sum_{c \in C'} \text{Tr}_{k(c)/k}(g_1(c) \otimes \cdots \otimes g_{n-1}(c) \otimes \partial_c(g_n, f)),$$

hence its image in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes by Definition 5.1. \square

11. MAIN THEOREM

11.1. Definition. Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$. We say that \mathcal{F} is *curve-like* if there exists an exact sequence in \mathbf{HI}_{Nis}

$$(11.1) \quad 0 \rightarrow T \rightarrow \mathcal{F} \rightarrow \bar{\mathcal{F}} \rightarrow 0$$

where $\bar{\mathcal{F}}$ is proper (Definition 10.1) and T sits in an exact sequence in \mathbf{HI}_{Nis}

$$(11.2) \quad 0 \rightarrow R_{E_1/k} \mathbf{G}_m \rightarrow R_{E_2/k} \mathbf{G}_m \rightarrow T \rightarrow 0$$

where E_1 and E_2 are étale k -algebras.⁵

This terminology is justified by the following lemma:

11.2. Lemma. *a) If C is a smooth curve over k , then $h_0^{\text{Nis}}(C)$ is the Nisnevich sheaf associated to the presheaf of relative Picard groups*

$$U \mapsto \text{Pic}(\bar{C} \times U, D \times U)$$

where \bar{C} is the smooth projective completion of C , $D = \bar{C} \setminus C$ and U runs through smooth k -schemes.

b) If X is a smooth projective variety over k , then, for any smooth variety U over k , we have

$$(11.3) \quad h_0^{\text{Nis}}(X)(U) = CH_0(X_{k(U)}),$$

where $k(U)$ denotes the total ring of fractions of U . In particular, $h_0^{\text{Nis}}(X)$ is birational.

c) For any smooth curve C , $h_0^{\text{Nis}}(C)$ is curve-like.

Proof. a) and b) are proven in [29, Th. 3.1] and in [7, Th. 2.2] respectively. We prove c). This follows from b) if C is projective over k . We assume C is affine. With the notation of a), we have the Gysin exact triangle [32, Prop. 3.5.4]

$$M(D)(1)[1] \rightarrow M(C) \rightarrow M(\bar{C}) \xrightarrow{+1}.$$

By [32, Th. 3.4.2], we have $h_i^{\text{Nis}}(C) = 0$ for all $i \neq 0$ and $h_1^{\text{Nis}}(\bar{C}) = R_{E/k} \mathbf{G}_m$, where $E = H^0(\bar{C}, \mathcal{O}_{\bar{C}})$. Hence we get an exact sequence

$$0 \rightarrow R_{E/k} \mathbf{G}_m \rightarrow R_{D/k} \mathbf{G}_m \rightarrow h_0^{\text{Nis}}(C) \rightarrow h_0^{\text{Nis}}(\bar{C}) \rightarrow 0,$$

which proves c). □

11.3. Remark. Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ be curve-like. The sheaves T and $\bar{\mathcal{F}}$ in (11.1) are uniquely determined by \mathcal{F} up to unique isomorphism. Indeed, this amounts to showing that any morphism $T \rightarrow \bar{\mathcal{F}}$ is trivial. This is reduced to the case $T = R_{E/k} \mathbf{G}_m$ as in (11.2), and further to $T = \mathbf{G}_m$ by adjunction as in Lemma 8.8. Then we have $\text{Hom}_{\mathbf{HI}_{\text{Nis}}}(\mathbf{G}_m, \bar{\mathcal{F}}) \simeq \bar{\mathcal{F}}_{-1}(k) = 0$ by definition (see (4.1) and Definition 10.1).

⁵It then follows from Hilbert's theorem 90 applied to $R_{E_1/k} \mathbf{G}_m$ that $T = T_{\text{ét}}$, hence that T agrees with the cokernel of $R_{E_1/k} \mathbf{G}_m \rightarrow R_{E_2/k} \mathbf{G}_m$ as tori; we shall not need this remark in the sequel.

We call T and $\bar{\mathcal{F}}$ the *toric* and *proper* part of \mathcal{F} respectively (see footnote 5).

11.4. Lemma. *a) Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ be curve-like with toric part T , and let C be a smooth proper connected k -curve. Let Z be a proper closed subset of C , $A = \mathcal{O}_{C,Z}$ and $K = k(C)$. Then the sequence*

$$0 \rightarrow T(A) \xrightarrow{f} T(K) \oplus \mathcal{F}(A) \xrightarrow{g} \mathcal{F}(K) \rightarrow 0$$

is exact, where f and g are given by $f(a) = (a, a)$ and $g(b, c) = b - c$, under the identification $T(A) \subset \mathcal{F}(A) \subset \mathcal{F}(K)$ and $T(A) \subset T(K) \subset \mathcal{F}(K)$.

b) Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$ be curve-like with toric parts T_1, \dots, T_n , and let C, Z, A, K be as in a). Then the group $\mathcal{F}_1(K) \otimes \dots \otimes \mathcal{F}_n(K)$ has the following presentation:

Generators: *for each subset $I \subseteq \{1, \dots, n\}$, elements $[I; f_1, \dots, f_n]$ with $f_i \in \mathcal{F}_i(A)$ if $i \in I$ and $f_i \in T_i(K)$ if $i \notin I$.*

Relations:

- *Multilinearity:*

$$[I; f_1, \dots, f_i + f'_i, \dots, f_n] = [I; f_1, \dots, f_i, \dots, f_n] + [I; f_1, \dots, f'_i, \dots, f_n].$$

- *Let $I \subsetneq \{1, \dots, n\}$ and let $i_0 \notin I$. Let $[I; f_1, \dots, f_n]$ be a generator. Suppose that $f_{i_0} \in T_{i_0}(A)$. Then $[I; f_1, \dots, f_n] = [I \cup \{i_0\}; f_1, \dots, f_n]$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T(A) & \longrightarrow & \mathcal{F}(A) & \longrightarrow & \bar{\mathcal{F}}(A) \longrightarrow 0 \\
& & \downarrow a & & \downarrow b & & \downarrow c \\
0 & \longrightarrow & T(K) & \longrightarrow & \mathcal{F}(K) & \longrightarrow & \bar{\mathcal{F}}(K) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus T_{-1}(k_i) & \longrightarrow & \bigoplus \mathcal{F}_{-1}(k_i) & \longrightarrow & \bigoplus \bar{\mathcal{F}}_{-1}(k_i) \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Here the k_i 's run through the residue fields of points of Z and the (exact) vertical sequences are those from [31, Th. 4.37]. Since $0 \rightarrow T \rightarrow \mathcal{F} \rightarrow \bar{\mathcal{F}} \rightarrow 0$ is an exact sequence of Nisnevich sheaves and any field has Nisnevich cohomological dimension zero, all horizontal

sequences are left exact and the middle one is also exact at $\bar{\mathcal{F}}(K)$. (See §4.1 for the exactness of the bottom row.) Since $\bar{\mathcal{F}}$ is proper, we have $\bar{\mathcal{F}}_{-1}(k_i) = 0$ (see the end of Remark 11.3); it follows that c is an isomorphism and that the upper horizontal sequence is also exact at $\bar{\mathcal{F}}(A)$. Now a) follows from a diagram chase and b) follows from a). \square

11.5. *Remark.* A shorter but more delicate proof is that the maps a, b, c have compatible retractions. Since C is a curve, this may be deduced from the proof of [32, Lemma 4.5] (see also [32, Cor. 4.18]).

11.6. **Proposition.** *Let C/k be a smooth proper connected curve, and let $v \in C, K = k(C)$. Then there exists a unique law associating to a system $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ of n curve-like sheaves a homomorphism*

$$\partial_v : \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* \rightarrow \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n)$$

such that

(i) *If σ is a permutation of $\{1, \dots, n\}$, the diagram*

$$\begin{array}{ccc} \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n) \\ \sigma \downarrow & & \sigma \downarrow \\ \mathcal{F}_{\sigma(1)}(K) \otimes \cdots \otimes \mathcal{F}_{\sigma(n)}(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_{\sigma(1)}, \dots, \mathcal{F}_{\sigma(n)}) \end{array}$$

commutes.

(ii) *If $[I, f_1, \dots, f_n]$ is a generator of $\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K)$ as in Lemma 11.4 b) for some Z containing v , with $I = \{1, \dots, i\}$, then*

$$\partial_v(f_1 \otimes \cdots \otimes f_n \otimes f) = \{f_1(v), \dots, f_i(v), \partial_v(\{f_{i+1}, \dots, f_n, f\}_{K/K})\}_{k(v)/k}$$

where $\partial_v(\{f_{i+1}, \dots, f_n, f\}_{K/K})$ is the residue (8.4).

Proof. By Lemma 11.4 b), it suffice to check that ∂_v agrees on relations. Up to permutation, we may assume $I = \{1, \dots, i\}$ and $i_0 = i + 1$. The claim then follows from Proposition 4.10. \square

11.7. **Lemma.** *a) Keep the notation of Proposition 11.6. Let L/K be a finite extension; write D for the smooth projective model of L and $h : D \rightarrow C$ for the corresponding morphism. Let $Z = h^{-1}(v)$. Write*

$\mathcal{F}_{n+1} = \mathbf{G}_m$. Then, for any $i \in \{1, \dots, n+1\}$, the diagram

$$\begin{array}{ccc}
 \mathcal{F}_1(L) \otimes \cdots \otimes \mathcal{F}_{n+1}(L) & \xrightarrow{(\partial_w)} & \bigoplus_{w \in Z} \tilde{K}(k(w); \mathcal{F}_1, \dots, \mathcal{F}_n) \\
 \uparrow u & & \downarrow (\mathrm{Tr}_{k(w)/k(v)}) \\
 \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_i(L) \otimes \cdots \otimes \mathcal{F}_{n+1}(K) & & \\
 \downarrow d & & \\
 \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_{n+1}(K) & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n)
 \end{array}$$

commutes, where u is given componentwise by functoriality for $j \neq i$ and by the identity for $j = i$, and d is given componentwise by the identity for $j \neq i$ and by $\mathrm{Tr}_{L/K}$ for $j = i$.

b) The homomorphisms ∂_v induce residue maps

$$\partial_v : \left(\mathcal{F}_1^{\otimes M} \otimes \cdots \otimes \mathcal{F}_n^{\otimes M} \otimes \mathbf{G}_m \right) (K) \rightarrow \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n).$$

which verify the compatibility of Lemma 4.2 b).

Proof. a) For clarity, we distinguish two cases: $i < n+1$ and $i = n+1$. In the former case, up to permutation we may assume $i = n$. It is enough to check commutativity on generators in the style of Lemma 11.4 b). Let T_l denote the toric part of \mathcal{F}_l . In view of Lemma 11.4 a) and Proposition 11.6 (i), it suffices to check the commutativity for $x = f_1 \otimes \cdots \otimes f_n \otimes f$ when one of the following two conditions is satisfied:

- (i) for some $j \in \{0, \dots, n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ ($1 \leq l \leq j$), $f_l \in T_l(K)$ ($j+1 \leq l \leq n-1$), $f_n \in T_n(L)$ and $f \in K^*$.
- (ii) for some $j \in \{0, \dots, n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ ($1 \leq l \leq j$), $f_l \in T_l(K)$ ($j+1 \leq l \leq n-1$), $f_n \in \mathcal{F}_n(\mathcal{O}_{D,Z})$ and $f \in K^*$.

Let $w \in Z$. If (i) holds, we have

$$\partial_w(u(x)) = \{f_1(w), \dots, f_j(w), \partial_w(\{f_{j+1}, \dots, f_n, f\}_{L/L})\}_{k(w)/k(w)}$$

and

$$\partial_v(d(x)) = \{f_1(v), \dots, f_j(v), \partial_v(\{f_{j+1}, \dots, \mathrm{Tr}_{L/K}(f_n), f\}_{K/K})\}_{k(v)/k(v)}.$$

Observe that the restriction of $f_l(v)$ to $k(w)$ is $f_l(w)$ for every $w \in Z$ and $l = 1, \dots, j$. Since the residue maps (∂_w) (8.4) verify the compatibility of Lemma 4.2, the commutativity for x follows. (Recall that $\mathrm{Tr}_{k(w)/k(v)}(\{a_1, \dots, a_n\}_{k(w)/k(w)}) = \{a_1, \dots, a_n\}_{k(w)/k(v)}$.)

If (ii) holds, we have

$$\partial_w(u(x)) = \{f_1(w), \dots, f_j(w), \partial_w(\{f_{j+1}, \dots, f_{n-1}, f\}_{L/L}), f_n(w)\}_{k(w)/k(w)}$$

and

$$\partial_v(d(x)) = \{f_1(v), \dots, f_j(v), \partial_v(\{f_{j+1}, \dots, f_{n-1}, f\}_{K/K}), \mathrm{Tr}_{L/K}(f_n)(v)\}_{k(v)/k(v)}.$$

In addition to the observation mentioned in (i), we remark that the restriction of $\partial_v(\{f_{j+1}, \dots, f_{n-1}, f\}_{K/K})$ to $k(w)$ is $\partial_w(\{f_{j+1}, \dots, f_{n-1}, f\}_{L/L})$ for every $w \in Z$. The commutativity for x follows from Lemma 4.2 b) applied to \mathcal{F}_n .

If $i = n + 1$ the check is similar, the projection formula working on the last variable.

Now b) follows from a) and the definition of \otimes^M recalled in §2.8. \square

11.8. Lemma. *The homomorphisms ∂_v of Lemma 11.7 induce residue maps*

$$\partial_v : \tilde{K}(K; \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m) \rightarrow \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n).$$

which verify the compatibility of Lemma 4.2 b).

Proof. Set $\mathcal{F}_{n+1} = \mathbf{G}_m$. Let $i < j$ be two elements of $\{1, \dots, n+1\}$ and let $\chi_i : \mathbf{G}_m \rightarrow \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \rightarrow \mathcal{F}_j$ be two cocharacters. Let $f \in K^* - \{1\}$. We must show that ∂_v vanishes on

$$x = f_1 \otimes \dots \otimes \chi_i(f) \otimes \dots \otimes \chi_j(1-f) \otimes \dots \otimes f_{n+1}$$

for any $(f_1, \dots, f_{n+1}) \in \mathcal{F}_1(K) \times \dots \times \mathcal{F}_{n+1}(K)$ (product excluding (i, j)). By functoriality, we may assume that χ_i, χ_j are the identity cocharacters. We distinguish two cases for clarity: $j < n+1$ and $j = n+1$. But exactly the same argument works for both cases. Presently we suppose $j < n+1$.

Up to permutation, we may assume $i = n-1$, $j = n$. Let us say that an element $(x_1, \dots, x_{n-2}) \in \mathcal{F}_1(K) \times \dots \times \mathcal{F}_{n-2}(K)$ is *in normal form* if, for each $s = 1, \dots, n-2$, either $x_s \in \mathcal{F}_s(\mathcal{O}_v)$ or $x_s \in T_s(K)$. (Here T_s is the toric part of \mathcal{F}_s .) Then Lemma 11.4 reduces us to the case where (f_1, \dots, f_{n-2}) is in normal form. Up to permutation, we may assume that $f_s \in \mathcal{F}_s(\mathcal{O}_v)$ for $s \leq r$ and $f_s \in T_s(K)$ for $r < s \leq n-2$. Then

$$\partial_v x = \{f_1(v), \dots, f_r(v), \partial_v(\{f_{r+1}, \dots, f_{n-2}, f, (1-f), f_{n+1}\}_{K/K})\}_{k(v)/k(v)}.$$

Let $\varphi_v : \tilde{K}(k(v), T_{r+1}, \dots, T_n) \rightarrow \tilde{K}(k(v), \mathcal{F}_1, \dots, \mathcal{F}_n)$ be the homomorphism induced by $(f_1(v), \dots, f_r(v))$ via (8.2), and let $\varphi_K : T_{r+1}(K) \otimes$

$\cdots \otimes T_n(K) \otimes K^* \rightarrow \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^*$ be the analogous homomorphism defined by (f_1, \dots, f_r) . The diagram

$$\begin{array}{ccc} T_{r+1}(K) \otimes \cdots \otimes T_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); T_{r+1}, \dots, T_n) \\ \varphi_K \downarrow & & \varphi_v \downarrow \\ \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n) \end{array}$$

commutes. But the top map factors through

$$\partial_v : \tilde{K}(K; T_{r+1}, \dots, T_n, \mathbf{G}_m) \rightarrow \tilde{K}(k(v); T_{r+1}, \dots, T_n)$$

obtained in (8.4), hence the desired vanishing.

Thus we have shown that the map ∂_v of Proposition 11.6 vanishes on $St(K; \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. The conclusion now follows from Lemma 11.7 b). \square

11.9. Let $\mathcal{F} \in \mathbf{HI}_{\text{Nis}}$ and let C be a smooth proper k -curve. The *support* of a section $f \in \mathcal{F}(k(C))$ is the finite set

$$\text{Supp}(f) = \{c \in C \mid f \notin \mathcal{F}(\mathcal{O}_{C,c})\}.$$

The following lemma and proposition generalize Lemma 7.4:

11.10. **Lemma.** *Let T_1, \dots, T_r be r curve-like tori. Put $T_{r+1} = \mathbf{G}_m$. Let D be a smooth proper k -curve and $Z \subset D$ a proper closed subset. If the field k is infinite, the group $\tilde{K}(k(D); T_1, \dots, T_r, \mathbf{G}_m)$ is generated by elements $\{f_1, \dots, f_{r+1}\}_{k(E)/k(D)}$ where E is another curve, $p : E \rightarrow D$ is a finite surjective morphism and $f_i \in T_i(k(E))$ satisfy*

$$(11.4) \quad \begin{aligned} \text{Supp}(f_i) \cap p^{-1}(Z) &= \emptyset \quad \text{for all } 1 \leq i \leq r, \\ \text{Supp}(f_i) \cap \text{Supp}(f_j) &= \emptyset \quad \text{for all } 1 \leq i < j \leq r. \end{aligned}$$

Proof. As in the proof of Proposition 8.10 b), we are reduced to the case where all T_i are $R_{E_i/k} \mathbf{G}_m$ for some étale k -algebras E_i/k . Using the formula

$$(R_{E_1/k} \mathbf{G}_{m,E_1})_{E_2} \simeq R_{E_1 \otimes_k E_2/E_2} \mathbf{G}_{m,E_1 \otimes E_2}$$

and Lemma 8.8 repeatedly, we are further reduced to the case all T_i are \mathbf{G}_m . Then it follows from Lemma 7.4. \square

11.11. **Proposition.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be n curve-like sheaves, and let C be a smooth proper k -curve. Put $\mathcal{F}_{n+1} = \mathbf{G}_m$. If the field k is infinite, the group $\tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$ is generated by elements $\{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$ where D is another curve, $D \rightarrow C$ is a finite surjective morphism and $f_i \in \mathcal{F}_i(k(D))$ satisfy*

$$(11.5) \quad \text{Supp}(f_i) \cap \text{Supp}(f_j) = \emptyset \quad \text{for all } 1 \leq i < j \leq n.$$

Proof. Let T_i be the toric part of \mathcal{F}_i . Given a finite surjective morphism $D \rightarrow C$ and $f_i \in \mathcal{F}_i(k(D))$ ($i = 1, \dots, n+1$), we construct a sequence $(Z_i, g_i^{(1)}, g_i^{(2)})_{i=1, \dots, n+1}$ of closed subsets $Z_i \subset D$ and sections $g_i^{(1)} \in \mathcal{F}_i(\mathcal{O}_{D, Z_i})$, $g_i^{(2)} \in T_i(k(D))$ such that $f_i = g_i^{(1)} + g_i^{(2)}$ by induction. First we put $Z_1 = \emptyset$, $g_1^{(1)} = f_1$ and $g_1^{(2)} = 0$. Suppose we have constructed $(Z_{i-1}, g_{i-1}^{(1)}, g_{i-1}^{(2)})$. We define $Z_i = Z_{i-1} \cup \text{Supp}(g_{i-1}^{(1)})$. Then we apply Lemma 11.4 a) to find $g_i^{(1)} \in \mathcal{F}_i(\mathcal{O}_{D, Z_i})$ and $g_i^{(2)} \in T_i(k(D))$ such that $f_i = g_i^{(1)} + g_i^{(2)}$. By construction, we have

$$\text{Supp}(g_i^{(1)}) \cap \text{Supp}(g_j^{(1)}) = \emptyset$$

for all $1 \leq i < j \leq n+1$, and

$$\{f_1, \dots, f_{n+1}\}_{k(D)/k(C)} = \sum_{\mathbf{e} \in \{1, 2\}^n} \{g_1^{(e_1)}, \dots, g_{n+1}^{(e_{n+1})}\}_{k(D)/k(C)},$$

where $\mathbf{e} = (e_1, \dots, e_{n+1})$. Given $\mathbf{e} \in \{1, 2\}^n$, let $I = \{i \in \{1, \dots, n\} \mid e_i = 1\}$: the collection of $g_i^{(1)}$ for $i \in I$ defines a homomorphism

$$\tilde{K}(k(D); T_{i_1}, \dots, T_{i_m}, \mathbf{G}_m) \rightarrow \tilde{K}(k(D); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$$

where $i_1 < \dots < i_m$ are the elements of $\{1, \dots, n\} \setminus I$. The proposition then follows by applying Lemma 11.10 with $Z = \cup_{e_i=1} \text{Supp}(g_i^{(1)})$ for each \mathbf{e} . \square

11.12. Lemma. *Let $C, D, \mathcal{F}_1, \dots, \mathcal{F}_n$ be as in Proposition 11.11. Let $f_i \in \mathcal{F}_i(k(D))$ and $v \in D$. Put $\xi := \{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$, regarded as an element of $\tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$.*

- (1) *If $v(f_{n+1} - 1) > 0$, then we have $\partial_v(\xi) = 0$.*
- (2) *Suppose (11.5) holds. If $v \in \text{Supp}(f_i)$ for some $1 \leq i \leq n$, then we have*

$$\partial_v(\xi) = \{f_1(v), \dots, \partial_v(f_i, f_{n+1}), \dots, f_n(v)\}_{k(v)/k}.$$

Proof. This follows from Corollary 4.11 and Proposition 4.10. \square

11.13. Proposition. *Let C be a smooth projective connected curve, and let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\text{Nis}}$ be curve-like. The composition*

$$\begin{aligned} \sum_{v \in C} \text{Tr}_{k(v)/k} \circ \partial_v : \tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m) &\rightarrow \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \\ &\rightarrow K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \end{aligned}$$

is the zero-map.

Proof. a) Assume first k infinite. If $\xi = \{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$ satisfies (11.5), then we have $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\xi) = 0$ by Definition 5.1 and Lemma 11.12 (2). Hence the claim follows from Proposition 11.11.

b) If k is finite, we use a classical trick: let p_1, p_2 be two distinct prime numbers, and let k_i be the \mathbf{Z}_{p_i} -extension of k . Let $x \in \tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. By a), the image of x in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k_1; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and $K(k_2; \mathcal{F}_1, \dots, \mathcal{F}_n)$, hence is 0 by a transfer argument. \square

Finally, we arrive at:

11.14. Theorem. *The homomorphism (1.1) is an isomorphism for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.*

Proof. It suffices to show the statement in Proposition 9.1 c). With the notation therein, \mathcal{A} is curve-like by Lemma 11.2 c). The image of (9.1) in $K(k; \mathcal{A}, \dots, \mathcal{A})$ is seen to be $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\{\iota, \dots, \iota, f\}_{k(C)/k(C)})$ by Lemma 11.12, hence trivial by Proposition 11.13. \square

12. APPLICATION TO ALGEBRAIC CYCLES

12.1. We assume k is of characteristic zero. Let X be a k -scheme of finite type, and let $M^c(X) := C_*^c(X) \in \mathbf{DM}_-^{\mathrm{eff}}$ be the motive of X with compact supports [32, §4.1]. Then the sheaf $\underline{CH}_0(X)$ of §1.4 agrees with $H_0(M^c(X))$ by [7, Th. 2.2]. If X is quasi-projective, we have an isomorphism

$$CH_{-i}(X, j + 2i) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M^c(X))(i)[-j]$$

for all $i \in \mathbf{Z}_{\geq 0}, j \in \mathbf{Z}$ by [32, Prop. 4.2.9].⁶

Proof of Theorem 1.5. Using Lemma 3.3, we see

$$\begin{aligned} & \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \underline{CH}_0(X_1)[0] \otimes \cdots \otimes \underline{CH}_0(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}) \\ & \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M^c(X_1) \otimes \cdots \otimes M^c(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}). \\ & \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, M^c(X)(r)[r]) \simeq CH_{-r}(X, r). \end{aligned}$$

(Here we used $\mathbf{G}_m[0] \simeq \mathbf{Z}(1)[1]$.) Now the theorem follows from Theorem 11.14. \square

⁶The proof of *loc. cit.* is written for equidimensional schemes but is the same in general. Moreover, the assumption “quasi-projective” can be removed if one replaces higher Chow groups by the Zariski hypercohomology of the cycle complex as in [14, after Theorem 1.7].

12.2. Let X be a k -scheme of finite type. Recall that for $i \in \mathbf{Z}_{\geq 0}, j \in \mathbf{Z}$ the motivic homology of X is defined by [6, Def. 9.4]

$$(12.1) \quad H_j(X, \mathbf{Z}(-i)) := \mathrm{Hom}_{\mathbf{DM}_{\underline{e}}^{\mathrm{eff}}}(\mathbf{Z}, M(X)(i)[-j]).$$

When $i = 0$, $H_j(X, \mathbf{Z}(0))$ agrees with Suslin homology [29].

12.3. **Theorem.** *Let X_1, \dots, X_n be k -schemes of finite type. Suppose either the X_i are smooth or $\mathrm{char} k = 0$. Put $X = X_1 \times \dots \times X_n$. For any $r \geq 0$, we have an isomorphism*

$$K(k; h_0^{\mathrm{Nis}}(X_1), \dots, h_0^{\mathrm{Nis}}(X_n), \mathbf{G}_m, \dots, \mathbf{G}_m) \xrightarrow{\sim} H_{-r}(X, \mathbf{Z}(-r)).$$

Proof. Using Lemma 3.3, we see

$$\begin{aligned} & \mathrm{Hom}_{\mathbf{DM}_{\underline{e}}^{\mathrm{eff}}}(\mathbf{Z}, h_0^{\mathrm{Nis}}(X_1)[0] \otimes \dots \otimes h_0^{\mathrm{Nis}}(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}) \\ & \simeq \mathrm{Hom}_{\mathbf{DM}_{\underline{e}}^{\mathrm{eff}}}(\mathbf{Z}, M(X_1) \otimes \dots \otimes M(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}). \\ & \simeq \mathrm{Hom}_{\mathbf{DM}_{\underline{e}}^{\mathrm{eff}}}(\mathbf{Z}, M(X)(r)[r]) \simeq H_{-r}(X, \mathbf{Z}(-r)). \end{aligned}$$

Now the theorem follows from Theorem 11.14. \square

12.4. *Remark.* If X_1, \dots, X_n are smooth projective varieties, then (1.3) is valid in any characteristic. Indeed, we have $M(X_i) = M^c(X_i)$ and hence $\underline{CH}_0(X_i) = h_0^{\mathrm{Nis}}(X_i)$. Moreover, [33] and [7, Appendix B] show $H_{-r}(X, \mathbf{Z}(-r)) \simeq CH_{-r}(X, r)$. Thus (1.3) follows from Theorem 12.3.

APPENDIX A. EXTENDING MONOIDAL STRUCTURES

A.1. Let \mathcal{A} be an additive category. We write $\mathrm{Mod}\text{-}\mathcal{A}$ for the category of contravariant additive functors from \mathcal{A} to abelian groups. This is a Grothendieck abelian category. We have the additive Yoneda embedding

$$y_{\mathcal{A}} : \mathcal{A} \rightarrow \mathrm{Mod}\text{-}\mathcal{A}$$

sending an object to the corresponding representable functor.

A.2. An object of $\mathrm{Mod}\text{-}\mathcal{A}$ is *free* if it is a direct sum of representable objects. Let $M \in \mathrm{Mod}\text{-}\mathcal{A}$: for any $A \in \mathcal{A}$, the Yoneda isomorphism

$$M(A) \simeq \mathrm{Mod}\text{-}\mathcal{A}(y_{\mathcal{A}}(A), M)$$

realises M canonically as a quotient of a free module:

$$L_0(M) = \bigoplus_{(A,f)} y_{\mathcal{A}}(A) \twoheadrightarrow M$$

where (A, f) runs through pairs of an object $A \in \mathcal{A}$ and an element $f \in M(A)$. Iterating, we get a canonical and functorial free resolution

$$(A.1) \quad \dots \rightarrow L_n(M) \rightarrow \dots \rightarrow L_0(M) \rightarrow M \rightarrow 0$$

as in [16, Lemma 8.1].

A.3. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. We have an induced functor $f^* : \text{Mod-}\mathcal{B} \rightarrow \text{Mod-}\mathcal{A}$ (“composition with f ”). As in [SGA4, Exp. 1, Prop. 5.1 and 5.4], the functor f^* has a left adjoint $f_!$ and a right adjoint f_* and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \text{Mod-}\mathcal{A} \\ f \downarrow & & f_! \downarrow \\ \mathcal{B} & \xrightarrow{y_{\mathcal{B}}} & \text{Mod-}\mathcal{B} \end{array}$$

is naturally commutative.

A.4. If f is fully faithful, then $f_!$ and f_* are fully faithful and f^* is a localisation, as in [SGA4, Exp. 1, Prop. 5.6].

A.5. Suppose that f has a left adjoint g . Then we have natural isomorphisms

$$g^* \simeq f_!, \quad g_* \simeq f^*$$

as in [SGA4, Exp. 1, Prop. 5.5].

A.6. Suppose further that f is fully faithful. Then $g^* \simeq f_!$ is fully faithful. From the composition

$$g^* g_* \Rightarrow \text{Id}_{\text{Mod-}\mathcal{A}} \Rightarrow g^* g_!$$

of the unit with the counit, one then deduces a canonical morphism of functors

$$g_* \Rightarrow g_!.$$

A.7. Let \mathcal{A} and \mathcal{B} be two additive categories. Their *tensor product* is the category $\mathcal{A} \boxtimes \mathcal{B}$ whose objects are finite collections (A_i, B_i) with $(A_i, B_i) \in \mathcal{A} \times \mathcal{B}$, and

$$(\mathcal{A} \boxtimes \mathcal{B})((A_i, B_i), (C_j, D_j)) = \bigoplus_{i,j} \mathcal{A}(A_i, C_j) \otimes \mathcal{B}(B_i, D_j).$$

We have a “cross-product” functor

$$\boxtimes : \text{Mod-}\mathcal{A} \times \text{Mod-}\mathcal{B} \rightarrow \text{Mod-}(\mathcal{A} \boxtimes \mathcal{B})$$

given by

$$(M \boxtimes N)((A_i, B_i)) = \bigoplus_i M(A_i) \otimes N(B_i).$$

A.8. Let \mathcal{A} be provided with a biadditive bifunctor $\bullet : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. We may view \bullet as an additive functor $\mathcal{A} \boxtimes \mathcal{A} \rightarrow \mathcal{A}$. We may then extend \bullet to $\text{Mod-}\mathcal{A}$ by the composition

$$\text{Mod-}\mathcal{A} \times \text{Mod-}\mathcal{A} \xrightarrow{\boxtimes} \text{Mod-}(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\bullet} \text{Mod-}\mathcal{A}.$$

This is an extension in the sense that the diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{y_{\mathcal{A}} \times y_{\mathcal{A}}} & \text{Mod-}\mathcal{A} \times \text{Mod-}\mathcal{A} \\ \bullet \times \bullet \downarrow & & \bullet \downarrow \\ \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \text{Mod-}\mathcal{A} \end{array}$$

is naturally commutative.

If \bullet is monoidal (resp. monoidal symmetric), then its associativity and commutativity constraints canonically extend to $\text{Mod-}\mathcal{A}$.

A.9. As a composition of right exact functors, \bullet is right exact in the abelian category $\text{Mod-}\mathcal{A}$. The tensor product of two free modules (as in §A.2) is free. On the other hand, free objects have no reason to be flat in general (see caveat in [16, Rk. 8.6]). We shall see in Corollary A.15 that they are flat when \mathcal{A} is rigid.

A.10. For $M \in \text{Mod-}\mathcal{A}$, let $L_{\bullet}(M) \rightarrow M$ be its canonical free resolution of M from (A.1). If $N \in \text{Mod-}\mathcal{A}$ is another object, then the sequence

$$L_1(M) \bullet L_0(N) \oplus L_0(M) \bullet L_1(N) \rightarrow L_0(M) \bullet L_0(N) \rightarrow M \bullet N \rightarrow 0$$

is exact, yielding a presentation of $M \bullet N$ by free objects.

A.11. *Example.* If $\mathcal{A} = \mathbf{Cor}$, then $\text{Mod-}\mathcal{A} = \mathbf{PST}$. The free resolution $L_{\bullet}(M)$ of an object $M \in \mathbf{PST}$ is Voevodsky's resolution $\mathcal{L}(M)$ in [32, p. 206]: from §A.10, we recover his definition of $M \otimes N$ in *loc. cit.* or in [16, Def. 8.2]

A.12. Let \mathcal{A}, \mathcal{B} be two additive symmetric monoidal categories, and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be an additive symmetric monoidal functor. The above definition shows that the functor $f_! : \text{Mod-}\mathcal{A} \rightarrow \text{Mod-}\mathcal{B}$ is also symmetric monoidal.

A.13. In §A.8, let us write $\bullet_! = \int$ for clarity. Let $P \in \text{Mod-}(\mathcal{A} \boxtimes \mathcal{A})$. Then $\int P$ is the *left Kan extension of P along \bullet* in the sense of [15, X.3]. This gives a formula for $\int P$ as a *coend* (ibid., Theorem X.4.1); for $A \in \mathcal{A}$:

$$(A.2) \quad \int P(A) = \int^{(B, B')} \mathcal{A}(A, B \bullet B') \otimes P(B, B').$$

In particular:

A.14. **Proposition.** *Suppose \mathcal{A} rigid. Then (A.2) simplifies as*

$$\int P(A) = \int^B P(B, A \bullet B^*)$$

where B^* is the dual of $B \in \mathcal{A}$. In particular, if $P = M \boxtimes N$ for $M, N \in \text{Mod } -\mathcal{A}$, we have for $A \in \mathcal{A}$:

$$(A.3) \quad (M \bullet N)(A) = \int^B M(B) \otimes N(A \bullet B^*)$$

which describes $M \bullet N$ as a “convolution”. In particular, for $N = y_{\mathcal{A}}(C)$, $M \bullet y_{\mathcal{A}}(C)$ is given by the formula

$$(A.4) \quad (M \bullet y_{\mathcal{A}}(C))(A) = M(A \otimes C^*).$$

Proof. Applying (A.2) and rigidity, we have

$$\begin{aligned} \int P(A) &= \int^{(B, B')} \mathcal{A}(A, B \bullet B') \otimes P(B, B') \\ &= \int^{(B, B')} \mathcal{A}(A \bullet B^*, B') \otimes P(B, B') \\ &= \int^B P(B, A \bullet B^*) \end{aligned}$$

because in the third formula, the variable B' is dummy (this simplification is not in Mac Lane!).

We get (A.4) from (A.3), since

$$\begin{aligned} \int^B M(B) \otimes y_{\mathcal{A}}(C)(A \bullet B^*) &= \int^B M(B) \otimes \mathcal{A}(A \bullet B^*, C) \\ &\simeq \int^B M(B) \otimes \mathcal{A}(A \bullet C^*, B) = M(A \bullet C^*) \end{aligned}$$

since the variable B is dummy in the last-but-one term. \square

A.15. Corollary. *If \mathcal{A} is rigid, any free object of $\text{Mod } -\mathcal{A}$ is flat.*

Proof. An immediate consequence of (A.4). \square

A.16. We shall need a refinement of Formula (A.3). For this, we first have the probably well-known lemma, of which we include a proof for lack of reference (unfortunately it is not in Mac Lane’s book either).

A.17. Lemma (change of variables). *Let $\mathcal{C} \stackrel{R}{\underset{L}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors between small categories (L is left adjoint and R is right adjoint); moreover, let $T : \mathcal{D}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{X}$ be a functor, where small colimits are representable in \mathcal{X} . Then there is a canonical isomorphism*

$$\int^c T(Lc, c) \simeq \int^d T(d, Rd).$$

Proof. Let $\eta : Id_{\mathcal{C}} \Rightarrow RL$ and $\varepsilon : LR \Rightarrow Id_{\mathcal{D}}$ be the unit and the counit of the adjunction. Using ε we get a natural transformation

$$\varepsilon_{d,d'}^* = T(\varepsilon_d, 1) : T(d, Rd') \rightarrow T(LRd, Rd').$$

By the universal property of coends, this yields a morphism $\varphi : \int^d T(d, Rd) \rightarrow \int^c T(Lc, c)$. Using η , we similarly get a natural transformation $\eta_{*,c'}^c : T(Lc, c') \rightarrow T(Lc, RLc')$ and a morphism $\psi : \int^c T(Lc, c) \rightarrow \int^d T(d, Rd)$.

Write $X = \int^d T(d, Rd)$. Checking that $\psi \circ \varphi = 1$ amounts to checking that, for any $d_0 \in \mathcal{D}$, the composition

$$T(d_0, Rd_0) \xrightarrow{\varepsilon_{d_0,d_0}^*} T(LRd_0, Rd_0) \rightarrow \int^c T(Lc, c) \xrightarrow{\psi} X$$

equals the canonical map $\rho_{d_0} : T(d_0, Rd_0) \rightarrow X$. By definition, this composition is equal to

$$T(d_0, Rd_0) \xrightarrow{\varepsilon_{d_0,d_0}^*} T(LRd_0, Rd_0) \xrightarrow{\eta_{*,Rd_0}^{Rd_0,Rd_0}} T(LRd_0, RLd_0) \xrightarrow{\rho_{LRd_0}} X,$$

that is:

$$\rho_{LRd_0} \circ T(1, \eta_{Rd_0}) \circ T(\varepsilon_{d_0}, 1) = \rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) \circ T(1, \eta_{Rd_0}).$$

By the universal property of X , we have the identity

$$\rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) = \rho_{d_0} \circ T(1, R(\varepsilon_{d_0}))$$

hence

$$\rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) \circ T(1, \eta_{Rd_0}) = \rho_{d_0} \circ T(1, R(\varepsilon_{d_0})) \circ T(1, \eta_{Rd_0}) = \rho_{d_0}$$

because of the adjunction identity $R(\varepsilon_{d_0}) \circ \eta_{d_0} = 1$. The proof that $\varphi \circ \psi = 1$ is similar. \square

A.18. *Example.* In proposition A.14., take $\mathcal{A} = \mathbf{ZSpan}(k)$: this category is rigid, all objects being self-dual (duality acts on morphisms by converting a span (f, g) into (g, f)). For any étale k -scheme X , the obvious forgetful functor $\omega : \mathbf{ZSpan}(X) \rightarrow \mathbf{ZSpan}(k)$ has a left adjoint $Y \mapsto X \times_k Y$. For a Mackey functor $M \in \mathbf{Mack}(k)$, write $M^X = M \circ \omega$. Applying Lemma A.17 with $\mathcal{C} = \mathbf{ZSpan}(k)$, $\mathcal{D} = \mathbf{ZSpan}(X)$ and $T(Y, Z) = N^X(Y) \otimes M(Z)$ and using that all objects are self-dual, we convert (A.3) into the formula

$$(M \otimes N)(X) = \int^{Y \in \mathbf{ZSpan}(X)} M^X(Y) \otimes N^X(Y).$$

Unfolding the definition of the coend, we immediately get the formula of §2.8. The case of more than two factors follows by associativity.

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