ERRATUM TO VOEVODSKY'S MOTIVES AND WEIL RECIPROCITY

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Abstract. We correct a mistake in §2.4 of the said paper.

In [1, §2.4], we wrote "The category $\mathbb{Z}\,\mathbf{Span}$ is isomorphic to the full subcategory of $\mathbf{Cor}$ consisting of smooth k-schemes of dimension 0". Tom Bachmann kindly pointed out to us that this statement is incorrect. Here we clarify the relationship between the two categories and show that it does not affect any argument about *cohomological* Mackey functors (the only Mackey functors appearing in [1]).

We retain the notation of [1].

1. Let $\mathbf{Cor}_0$ be the full subcategory of $\mathbf{Cor}$ given by 0-dimensional smooth schemes (= étale k-schemes). If $f : X \to Y$ is a surjective morphism of degree d of étale k-schemes, we have the formula in $\mathbf{Cor}_0$

$$ {}^t f \circ f = d. \tag{1}$$

2. There is a canonical functor

$$\varepsilon : \mathbb{Z}\,\mathbf{Span} \to \mathbf{Cor}_0 \tag{2}$$

which is the identity on objects and sends a span (2.1) from [1]

$$X \xleftarrow{g} Z \xrightarrow{f} Y \tag{3}$$

to $f \circ {}^t g$.

Lemma 3. *In* (3)*, assume Z irreducible and let $\bar{Z}$ be its image in $X \times Y$, viewed as an element of $\mathbf{Cor}_0(X, Y)$. Then $\varepsilon(f, g) = [Z : \bar{Z}]\bar{Z}$.*

Proof. This follows from the formula for the composition of finite correspondences. □

Proposition 4. *Let $M \in \mathbf{Mack}$ be a Mackey functor, i.e. an additive contravariant functor from $\mathbb{Z}\,\mathbf{Span}$ to $\mathbf{Ab}$. Then M is cohomological if and only if it factors through ε. This yields an equivalence*

$$\mathbf{Mack}_c \simeq \mathrm{Mod}-\mathbf{Cor}_0\,.$$

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Proof. If M factors through ε, it is cohomological thanks to (1). Conversely, if M is cohomological, consider a span (3) with Z irreducible, and let $\bar{Z}$ be as in Lemma 3. So we have a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[dd, "\pi"] & \\ X & & Y. \\ & \bar{Z} \arrow[ul, "\bar{g}"] \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Then $M^*(f) = M^*(\pi)M^*(\bar{f})$, $M_*(g) = M_*(\bar{g})M_*(\pi)$, thus

$$\begin{aligned} M(f,g) &= M_*(g)M^*(f) = M_*(\bar{g})M_*(\pi)M^*(\pi)M^*(\bar{f}) \\ &= \deg(\pi)M_*(\bar{g})M^*(\bar{f}) = \deg(\pi)M(\bar{f},\bar{g}) = M(\varepsilon(f,g)) \end{aligned}$$

by Lemma 3.

(Alternately, Proposition 4 follows from combining [4, Th. 4.3] and a version of [3, Prop. 3.4.1].) □

5. Proposition 4 justifies and corrects [1, 2.4]: the inclusion functor $\mathbf{Cor}_0 \hookrightarrow \mathbf{Cor}$ induces an exact functor

$$\rho : \mathbf{PST} \to \mathbf{Mack}_c$$

(in loc. cit., it is not necessary to restrict ρ to $\mathbf{HI}$ to get into $\mathbf{Mack}_c$).

6. To obtain [1, (2.9)], it remains to show that $\varepsilon^* : \mathbf{Mack}_c \to \mathbf{Mack}$ is symmetric monoidal with respect to the tensor structures induced by those of $\mathbb{Z}\,\mathbf{Span}$ and $\mathbf{Cor}_0$. (Recall these tensor structures: on objects they are given by the product of étale k-schemes; the tensor product of two spans (f,g) and (f',g') is $(f\times f', g\times g')$ and the tensor product of finite correspondences is the usual one.) This is obvious if k is algebraically closed, because ε is then a $\otimes$-isomorphism of $\otimes$-categories. The general case follows from

Proposition 7. *Let $\varepsilon : \mathcal{A} \to \mathcal{B}$ be a full $\otimes$-functor between rigid symmetric monoidal categories, which is the identity on objects. Then the natural morphism*

$$\varepsilon^* M \otimes_{\mathcal{A}} \varepsilon^* N \to \varepsilon^*(M \otimes_{\mathcal{B}} N) \tag{4}$$

is an isomorphism for any $M, N \in \mathrm{Mod}\text{–}\mathcal{B}$.

Before starting the proof, let us clarify the somewhat improper use of "dummy" in [1, last part of proof of A.14]:

Lemma 8. *Let $\mathcal{A}$ be an additive category, and let $M \in \operatorname{Mod}\text{–}\mathcal{A}$. Then there is a canonical isomorphism*

$$\theta : \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{A}(A,B) \xrightarrow{\sim} M(A)$$

for any $A \in \mathcal{A}$.

Proof. The "evaluation" morphisms $M(B) \otimes \mathcal{A}(A,B) \to M(A)$ mapping $m \otimes f$ to f^*m are linked by commutative diagrams like Diagram (3) of [2, p. 219]: this provides the map θ. Let us show that the map $\lambda : M(A) \to \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{A}(A,B)$ given by $\lambda(m) =$ (the class of) $m \otimes 1_A \in M(A) \otimes \mathcal{A}(A,A)$ is inverse to θ. It is obvious that $\theta\lambda$ is the identity. To check $\lambda\theta = \mathrm{id}$, take $B \in \mathcal{A}, m \in M(B)$ and $g \in \mathcal{A}(A,B)$. Then we have $\lambda\theta(m \otimes g) = g^*(m) \otimes 1_A = m \otimes g$, where the last equality holds because $m \otimes 1_A \in M(B) \otimes \mathcal{A}(A,A)$ is mapped to $m \otimes g \in M(B) \otimes \mathcal{A}(A,B)$ (resp. to $g^*(m) \otimes 1_A \in M(A) \otimes \mathcal{A}(A,A)$) by $1 \otimes g_*$ (resp. by $g^* \otimes 1$). □

We also have the following lemma:

Lemma 9. *Let $\varepsilon : \mathcal{A} \to \mathcal{B}$ be a functor, and let $T : \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to$ **Set** be a bifunctor. Then there is a canonical morphism $\int^{A\in\mathcal{A}} T(\varepsilon A, \varepsilon A) \to \int^{B\in\mathcal{B}} T(B,B)$. If ε surjective on objects, this is an surjection; if ε is moreover full and bijective on objects, this is a bijection.*

Proof. We may interpret coends as colimits by the dual of [2, Prop. 1 p. 224]. The first statement is then obvious (compare Formula (1) in [2, p. 217]), and the second one follows by inspection (the surjectivity of ε on objects gives surjectivity on generators, its bijectivity gives bijectivity on generators and its fullness gives surjectivity on relations). Alternatively, this can also be shown by using the final functor theorem of [2, p. 217, Th. 1]; details are left to the interested readers. □

10. We can now prove Proposition 7. As recalled in [1, A.3], ε^* has a right adjoint, hence commutes with arbitrary colimits. The two tensor products $\varepsilon^*M \otimes_{\mathcal{A}} -$ and $M \otimes_{\mathcal{B}} -$ also commute with arbitrary colimits, as seen from [1, A.10]. Thus we are reduced to the case where N is representable, say $N = y_{\mathcal{B}}(C)$ for $C \in \mathcal{B}$ (where $y_{\mathcal{B}} : \mathcal{B} \to \operatorname{Mod}\text{–}\mathcal{B}$ is the additive Yoneda embedding). For any $P \in \operatorname{Mod}\text{–}\mathcal{B}$ and any $A \in \mathcal{A}$, we have by definition

$$\varepsilon^*P(A) = P(\varepsilon A) = P(A)$$

since ε is the identity on objects. Using [1, (A.4)], this first yields

$$\varepsilon^*(M \otimes_{\mathcal{B}} y_{\mathcal{B}}(C))(A) = M(A \otimes_{\mathcal{B}} C^*),$$

where C^* is the dual of C. Using now [1, (A.3)], we compute

$$(\varepsilon^* M \otimes_{\mathcal{A}} \varepsilon^* y_{\mathcal{B}}(C))(A) = \int^{B\in\mathcal{A}} M(\varepsilon B) \otimes y_{\mathcal{B}}(C)(\varepsilon(A \otimes_{\mathcal{A}} B^*))$$

$$= \int^{B\in\mathcal{A}} M(\varepsilon B) \otimes y_{\mathcal{B}}(C)(\varepsilon A \otimes_{\mathcal{B}} (\varepsilon B)^*)) \quad \text{(monoidality of } \varepsilon)$$

$$= \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{B}(A \otimes_{\mathcal{B}} B^*, C)$$

$$= \int^{B\in\mathcal{A}} M(B) \otimes \mathcal{B}(A \otimes_{\mathcal{B}} C^*, B) \quad \text{(rigidity, compare [1, bottom p. 2791])}$$

$$= \int^{B\in\mathcal{B}} M(B) \otimes \mathcal{B}(A \otimes_{\mathcal{B}} C^*, B) \quad \text{(Lemma 9)}$$

$$= M(A \otimes_{\mathcal{B}} C^*) \quad \text{(Lemma 8)}.$$

With these identifications, it is clear that (4) becomes the identity map. □

11. To summarize this discussion: cohomological Mackey functors are exactly modules over $\mathbf{Cor}_0$; the relations on tensor product coming from the full transfer structure of Mackey functors are redundant as long as we work with cohomological Mackey functors.

12. Here are more minor errata:

- In the second diagram of §2.1, the arrows f^* and f'^* should point in the opposite direction.
- In the diagram in §A.8, the left (resp. right) vertical map should read $\bullet$ (resp. $\bullet_! \circ \boxtimes$).
- Throughout the appendix, references [2, Example 1] should read [2, Exposé 1].

References

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